Segmenting Multiple Time Series by Contemporaneous Linear Transformation

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Abstract

We seek for a contemporaneous linear transformation for a \( p \)-variate time series such that the transformed series is segmented into several lower-dimensional subseries, and those subseries are uncorrelated with each other both contemporaneously and serially. The method may be viewed as an extension of principal component analysis for multiple time series. Technically it also boils down to an eigenanalysis for a positive definite matrix. When \( p \) is large, an additional step is required to perform a permutation in terms of either maximum cross-correlations or FDR based on multiple tests. The asymptotic theory is established for both fixed \( p \) and diverging \( p \) when the sample size \( n \) tends to infinity, reflecting the fact that the method also applies when the dimension \( p \) is large in relation to \( n \). Numerical experiments with both simulated and real datasets indicate that the proposed method is an effective initial step in analysing multiple time series data, which leads to substantial dimension-reduction in modelling and forecasting high-dimensional linear dynamical structures. The method can also be adapted to segment multiple volatility processes.

Keywords: \( \alpha \)-mixing; Autocorrelation; Cross-correlation; Dimension reduction; Eigenanalysis; High-dimensional time series; Weak stationarity.

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1 Introduction

Modelling multiple time series, also called vector time series, is always a challenge, even when the vector dimension $p$ is moderately large. While most the inference methods and the associated theory for univariate autoregressive and moving average (ARMA) processes have found their multivariate counterparts (Lütkepohl, 2006), vector autoregressive and moving average (VARMA) models are seldom used directly in practice when $p \geq 3$. This is partially due to the lack of identifiability for VARMA models in general. More fundamentally those models are overparametrized; leading to flat likelihood functions which cause innate difficulties in statistical inference. Therefore finding an effective way to reduce the number of parameters is particularly felicitous in modelling and forecasting multiple time series. The urge for doing so is more pertinent in this modern information age, as it has become commonplace to access and to analyse high dimensional time series data with dimension $p$ in the order of hundreds or more. Big time series data arise from, among others, panel study for economic and natural (such as weather) phenomena, social network, healthcare and public health, financial market, supermarket transactions, information retrieval and recommender systems.

Available methods to reduce the number of parameters in modelling vector time series can be divided into two categories: regularization and dimension reduction. The former imposes some conditions on the structure of a VARMA model. The latter represents a high-dimensional process in terms of several lower-dimensional processes. Various regularization methods have been developed in literature. For example, Jakeman et al. (1980) adopted a two stage regression strategy based on instrumental variables to avoid using moving average explicitly. Different canonical structures are imposed on VARMA models [Chapter 3 of Reinsel (1993), and Chapter 4 of Tsay (2014), and references within]. Structural restrictions are imposed in order to specify and to estimate some reduced forms of vector autoregressive (VAR) models [Chapter 9 of Lütkepohl (2006), and references within]. Davis et al. (2012) proposed a VAR model with sparse coefficient matrices based on partial spectral coherence. Under different sparsity assumptions, VAR models have been estimated by LASSO regularization (Shojaie and Michailidis, 2010; Song and Bickel, 2011), or by the Dantzig selector (Han and Liu, 2013). Guo et al. (2014) consider high dimensional autoregression with banded coefficient matrices. The dimension reduction methods include the canonical correlation analysis of Box and Tiao (1977), the independent components analysis of Back and Weigend (1997), the principal components analysis (PCA) of Stock and Watson (2002), the scalar component analysis of Tiao and Tsay (1989) and Huang and Tsay (2014), the dynamic orthogonal components analysis and Matteson and Tsay (2011). Another popular idea is to represent multiple time series in terms of a few factors defined in various ways; see, among others, Peña and Box (1987), Stock and Watson (2005), Bai and Ng (2002), Forni et al. (2005), Pan and Yao (2008), Lam et al. (2011), Lam and Yao (2012) and Chang et al. (2013a).

A new dimension reduction method is proposed in this paper. We seek for a contemporaneous linear transformation such that the transformed series is segmented into several lower-dimensional subseries, and those subseries are uncorrelated with each other both contemporaneously and
Therefore they can be modelled or forecasted separately, as far as linear dependence is concerned. This reduces the number of parameters involved in depicting linear dynamic structure substantially. Technically the proposed method boils down to an eigenanalysis for a positive definite matrix which is a quadratic function of the cross correlation matrix function for the observed process. Hence it is easy to implement and the required computation can be carried out with, for example, an ordinary personal computer or laptop for the data with dimension $p$ in the order of thousands.

The method can be viewed as an extension of the standard PCA for multiple time series. However the segmented subseries are not guaranteed to exist as those subseries must not correlated with each other across all times. This is a marked difference from the standard PCA. The real data examples in Section 4.1 below indicate that it is often reasonable to assume that the segmentation exists. Furthermore, when the assumption is invalid, the proposed method provides some approximate segmentations which ignore some weak though significant correlations, and those weak correlations are of little practical use for modelling and forecasting. Thus the proposed method can be used as an initial step in analysing multiple time series, which often transforms a multi-dimensional problem into several lower-dimensional problems. Furthermore the results obtained for the transformed subseries can be easily transformed back to the original multiple time series. Illustration with real data examples also indicates clearly the advantages in post-sample forecasting from using the proposed segmentation method.

The rest of the paper is organised as follows. The methodology is spelled out in Section 2. The associated asymptotic properties are presented in Section 3 in which the three cases are considered: (i) $p$ is fixed, (ii) $p = o(n^c)$, and (iii) $\log p = o(n^c)$, as $n \to \infty$, where $c$ is some positive constant. Numerical illustration with both real and simulated data are reported in Section 4. Section 5 extends the method to segment a multiple volatility process into several lower-dimensional volatility sub-processes. All technical proofs are relegated into the Appendix.

We always use the following notation. For any $m \times k$ matrix $H = (h_{i,j})$, let
\[
\|H\|_2 = \sqrt{\lambda_{\text{max}}(HH')} \quad \text{and} \quad \|H\|_{\infty} = \max_{1 \leq i \leq m} \sum_{j=1}^k |h_{ij}|,
\]
where $\lambda_{\text{max}}(HH')$ denotes the maximum eigenvalue of $HH'$.

## 2 Methodology

### 2.1 Setting and method

Let $y_t$ be observable $p \times 1$ weakly stationary time series. We assume that $y_t$ admits a latent segmentation structure:
\[
y_t = Ax_t, \quad (2.1)
\]
where $x_t$ is an unobservable $p \times 1$ weakly stationary time series consisting of $q (> 1)$ both contemporaneously and serially uncorrelated subseries, and $A$ is an unknown constant matrix. Hence all
the autocovariance matrices of \( \mathbf{x}_t \) are of the same block-diagonal structure with \( q \) blocks. Without the loss of generality we may assume

\[
\text{Var}(\mathbf{y}_t) = \mathbf{I}_p \quad \text{and} \quad \text{Var}(\mathbf{x}_t) = \mathbf{I}_p, \tag{2.2}
\]

where \( \mathbf{I}_p \) denotes the \( p \times p \) identity matrix. This first equation in (2.2) amounts to replace \( \mathbf{y}_t \) by \( \hat{\mathbf{y}}_{t}^{-1/2} \mathbf{y}_t \) as a preliminary step in practice, where \( \hat{\mathbf{y}} \) is a consistent estimator for Var(\( \mathbf{y}_t \)). As both \( \mathbf{A} \) and \( \mathbf{x}_t \) are unobservable, the second equation in (2.2) implies that we view \( (\mathbf{A}\{\text{Var}(\mathbf{x}_t)\}^{1/2}, \{\text{Var}(\mathbf{x}_t)\}^{-1/2}\mathbf{x}_t) \) as \( (\mathbf{A}, \mathbf{x}_t) \) in (2.1). More importantly, the latter perspective will not alter the block-diagonal structure of the autocovariance matrices of \( \mathbf{x}_t \). Now it follows from (2.1) and (2.2) that

\[
\mathbf{I}_p = \text{Var}(\mathbf{y}_t) = \mathbf{A}\text{Var}(\mathbf{x}_t)\mathbf{A}' = \mathbf{A}\mathbf{A}'.
\]

Thus, \( \mathbf{A} \) in (2.1) is an orthogonal matrix under (2.2).

Denote the segmentation of \( \mathbf{x}_t \) by

\[
\mathbf{x}_t = \begin{pmatrix}
x_t^{(1)} \\
\vdots \\
x_t^{(q)}
\end{pmatrix}, \quad \text{where Cov}(x_t^{(i)}, x_t^{(j)}) = 0 \quad \text{for all } t, s \text{ and } i \neq j. \tag{2.3}
\]

Let \( p_j \) be the length of \( x_t^{(j)} \). Then \( \sum_j p_j = p \). Write \( \mathbf{A} = (\mathbf{A}_1, \ldots, \mathbf{A}_q) \), where \( \mathbf{A}_j \) has \( p_j \) columns. Since \( \mathbf{x}_t = \mathbf{A}'\mathbf{y}_t \), it holds that

\[
x_t^{(j)} = \mathbf{A}_j'\mathbf{y}_t, \quad j = 1, \ldots, q. \tag{2.4}
\]

Let \( \mathbf{H}_j \) be any \( p_j \times p_j \) orthogonal matrix, and \( \mathbf{H} = \text{diag}(\mathbf{H}_1, \ldots, \mathbf{H}_q) \). Then \( (\mathbf{A}, \mathbf{x}_t) \) in (2.1) can be replaced by \( (\mathbf{AH}, \mathbf{H}'\mathbf{x}_t) \) while (2.3) still holds. Hence \( \mathbf{A} \) and \( \mathbf{x}_t \) are not uniquely identified in (2.1), even with the additional assumption (2.2). In fact only \( \mathcal{M}(\mathbf{A}_1), \ldots, \mathcal{M}(\mathbf{A}_q) \) are uniquely defined by (2.1), where \( \mathcal{M}(\mathbf{A}_j) \) denotes the linear space spanned by the columns of \( \mathbf{A}_j \). Consequently, \( \mathbf{\Gamma}_j'\mathbf{y}_t \) can be taken as \( x_t^{(j)} \) for any \( p \times p_j \) matrix \( \mathbf{\Gamma}_j \) as long as \( \mathbf{\Gamma}_j'\mathbf{\Gamma}_j = \mathbf{I}_{p_j} \) and \( \mathcal{M}(\mathbf{\Gamma}_j) = \mathcal{M}(\mathbf{A}_j) \).

To discover the latent segmentation, we need to estimate \( \mathbf{A} = (\mathbf{A}_1, \ldots, \mathbf{A}_q) \), or more precisely, to estimate linear spaces \( \mathcal{M}(\mathbf{A}_1), \ldots, \mathcal{M}(\mathbf{A}_q) \). To this end, we introduce some notation first. For any integer \( k \), let

\[
\Sigma_y(k) = \text{Cov}(\mathbf{y}_{t+k}, \mathbf{y}_t) \quad \text{and} \quad \Sigma_x(k) = \text{Cov}(\mathbf{x}_{t+k}, \mathbf{x}_t).
\]

For a prescribed positive integer \( k_0 \), define

\[
\mathbf{W}_y = \sum_{k=0}^{k_0} \Sigma_y(k)\Sigma_y(k)' = \mathbf{I}_p + \sum_{k=1}^{k_0} \Sigma_y(k)\Sigma_y(k)', \tag{2.5}
\]

\[
\mathbf{W}_x = \sum_{k=0}^{k_0} \Sigma_x(k)\Sigma_x(k)' = \mathbf{I}_p + \sum_{k=1}^{k_0} \Sigma_x(k)\Sigma_x(k)'.
\]

Then both \( \Sigma_x(k) \) and \( \mathbf{W}_x \) are block-diagonal, and

\[
\mathbf{W}_y = \mathbf{A}\mathbf{W}_x\mathbf{A}'. \tag{2.6}
\]
Note that both $W_y$ and $W_x$ are positive definite matrices. Let

$$W_x\Gamma_x = \Gamma_x D,$$

(2.7)
i.e. $\Gamma_x$ is a $p \times p$ orthogonal matrix with the columns being the orthonormal eigenvectors of $W_x$, and $D$ is a diagonal matrix with the corresponding eigenvalues as the elements on the main diagonal. Then (2.6) implies that

$$W_y A \Gamma_x = A \Gamma_x D.$$

Hence the columns of $\Gamma_y \equiv A \Gamma_x$ are the orthonormal eigenvectors of $W_y$. Consequently,

$$\Gamma_y' y_t = \Gamma_x' A' y_t = \Gamma_x' x_t,$$

(2.8)
the last equality follows from (2.1). Put

$$W_x = \text{diag}(W_{x,1}, \ldots, W_{x,q}).$$

(2.9)

Then $W_{x,j}$ is a $p_j \times p_j$ positive definite matrix, and the eigenvalues of $W_{x,j}$ are also the eigenvalues of $W_x$. Suppose that $W_{x,i}$ and $W_{x,j}$ do not share the same eigenvalues for any $1 \leq i < j \leq q$. Then if we line up the eigenvalues of $W_x$ (i.e. the eigenvalues of $W_{x,1}, \ldots, W_{x,q}$ combining together) in the main diagonal of $D$ according to the order of the blocks in $W_x$, $\Gamma_x$ must be a block-diagonal orthogonal matrix of the same shape as $W_x$; see Proposition 1(i) below (and its proof in Appendix). However the order of the eigenvalues is latent, and any $\Gamma_x$ defined by (2.7) is nevertheless a column-permutation of such a block-diagonal orthogonal matrix; see Proposition 1(ii). Hence each component of $\Gamma_x' x_t$ is a linear transformation of the elements in one of the $q$ subseries only, i.e. the $p$ components of $\Gamma_y' y_t = \Gamma_x' x_t$ can be partitioned into the $q$ groups such that there exist neither contemporaneous nor serial correlations across different groups. Thus $\Gamma_y' y_t$ can be regarded as a permutation of $x_t$, and $\Gamma_y$ can be viewed as a column-permutation of $A$; see the discussion below (2.4). This leads to the following two-step estimation for $A$ and $x_t$:

**Step 1.** Let $\hat{S}$ be a consistent estimator for $W_y$. Calculate $\hat{\Gamma}_y$ which is a $p \times p$ orthogonal matrix with the columns being the orthonormal eigenvectors of $\hat{S}$.

**Step 2.** The columns of $\hat{A} = (\hat{A}_1, \ldots, \hat{A}_q)$ is a permutation of the columns of $\hat{\Gamma}_y$ such that $\hat{x}_t = \hat{A}' y_t$ is segmented into $q$ uncorrelated subseries $\hat{x}_t^{(j)} = \hat{A}_j' y_t$, $j = 1, \ldots, q$.

Step 1 is the key, as it provides an estimator for $A$ except that the columns of the estimator are not grouped together according to the latent segmentation. Its theoretical properties will be presented in Section 3. The permutation in Step 2 above can be carried out in principle by visual observation: plot cross correlogram of $\hat{z}_t \equiv \hat{\Gamma}_y' y_t$ (using, for example, R-function acf); see Figs. 4, 8 and 19 below. We then put those components of $\hat{z}_t$ together when there exist significant cross-correlations (at any lags) between those component series. Then $\hat{A}$ is obtained by re-arranging the order of the columns of $\hat{\Gamma}_y$ accordingly.
Remark 1. Appropriate precaution should be exercised in a visual observation stated above. First the visual observation become impractical when \( p \) is large. Furthermore most correlogram plots produced by statistical packages (including \( R \)) use the confidence bounds at \( \pm 1.96/\sqrt{n} \) for sample cross-correlations of two time series. Unfortunately those bounds are only valid if at least one of the two series is white noise. In general, the confidence bounds depend on the autocorrelations of the two series. See Theorem 7.3.1 of Brockwell and Davis (1996). In Section 2.2 below, we describe how the permutation can be performed analytically without the benefit of visual observation for the cross correlogram of \( \tilde{z}_t \).

Proposition 1. (i) The orthogonal matrix \( \Gamma_x \) in (2.7) can be taken as a block-diagonal orthogonal matrix with the same block structure as \( W_x \).

(ii) An orthogonal matrix \( \Gamma_x \) satisfies (2.7) if and only if its columns are a permutation of the columns of a block-diagonal orthogonal matrix described in (i), provided that any two blocks \( W_{x,i} \) and \( W_{x,j} \), \( 1 \leq i < j \leq q \), do not share the same eigenvalues.

Proposition 1(ii) requires the condition that the \( q \) blocks of \( W_x \) do not share the same eigenvalue(s). However it does not rule out the possibility that each block \( W_{x,j} \), \( 1 \leq j \leq q \), may have multiple eigenvalues. See the remark below Lemma 1 in Section 7.

2.2 Permutation

2.2.1 Permutation rule

The columns of \( \tilde{A} \) is a permutation of the columns of \( \tilde{\Gamma}_y \). The permutation is determined by grouping the components of \( \tilde{z}_t = \tilde{\Gamma}_y y_t \) into \( q \) groups, where \( q \) and the cardinal numbers of those groups are also unknown. We call two component series of \( \tilde{z}_t \) connected if the multiple null hypothesis

\[
H_0 : \rho(k) = 0 \quad \text{for any } k = 0, \pm 1, \pm 2, \ldots, \pm m
\]  (2.10)

is rejected, where \( \rho(k) \) denotes the cross correlation between the two component series at lag \( k \), and \( m \geq 1 \) is a prescribed integer. Thus there exists significant evidence indicating non-zero correlations between two connected component series. Hence those components should be put in the same group.

The permutation in Step 2 in Section 2.1 above can be performed as follows.

i. Start with the \( p \) groups with each group containing one component of \( \tilde{z}_t \) only.

ii. Combine two groups together if one connected pair are split over the two groups.

iii. Repeat Step ii above until all connected components are within one group.

We introduce below two methods for identifying the connected pair components of \( \tilde{z}_t = \tilde{\Gamma}_y y_t \).
2.2.2 Maximum cross correlation method

One natural way to test hypothesis $H_0$ in (2.10) is to use the maximum cross correlation over the lags between $-m$ and $m$. Write $\bar{z}_t = (\bar{z}_{1,t}, \ldots, \bar{z}_{p,t})'$. For any $1 \leq i < j \leq p$, the sample cross correlation of $\bar{z}_{i,t}$ and $\bar{z}_{j,t}$ is

$$
\hat{\rho}_{i,j}(h) = \begin{cases} 
\frac{\sum_{t=1}^{n-h}(\bar{z}_{i,t+h} - \bar{z}_{i, \cdot})(\bar{z}_{j,t} - \bar{z}_{j, \cdot})}{\sqrt{\sum_{t=1}^{n}(\bar{z}_{i,t} - \bar{z}_{i, \cdot})^2 \sum_{t=1}^{n}(\bar{z}_{j,t} - \bar{z}_{j, \cdot})^2}} & \text{if } h \geq 0; \\
\frac{\sum_{t=1}^{n-h}(\bar{z}_{i,t+h} - \bar{z}_{i, \cdot})(\bar{z}_{j,t} - \bar{z}_{j, \cdot})}{\sqrt{\sum_{t=1}^{n}(\bar{z}_{i,t} - \bar{z}_{i, \cdot})^2 \sum_{t=1}^{n}(\bar{z}_{j,t} - \bar{z}_{j, \cdot})^2}} & \text{if } h < 0;
\end{cases}
$$

(2.11)

where $\bar{z}_{i, \cdot} = n^{-1} \sum_{t=1}^{n} \bar{z}_{i,t}$. The maximum cross correlation statistic is defined as

$$
\hat{L}_n(i, j) = \max_{|h| \leq m} |\hat{\rho}_{i,j}(h)|.
$$

(2.12)

We would reject $H_0$ for the pair $(\bar{z}_{i,t}, \bar{z}_{j,t})$ if $\hat{L}_n(i, j)$ is greater than an appropriate threshold value.

Instead of conducting multiple tests for each of the $p_0 = p(p-1)/2$ pairs components of $\bar{z}_t$, we propose a ratio-based statistic to single out those pairs for which $H_0$ will be rejected. To this end, we re-arrange the $p_0$ maximum cross correlations $\hat{L}_n(i, j), 1 \leq i < j \leq p$, in the descending order: $\hat{L}_1 \geq \cdots \geq \hat{L}_{p_0}$. Define

$$
\hat{r} = \arg \max_{1 \leq j < c_0 p_0} \hat{L}_j/\hat{L}_{j+1},
$$

(2.13)

where $c_0 \in (0, 1)$ is a prescribed constant. In all the numerical examples in Section 4 below we use $c_0 = 0.75$. We reject $H_0$ for the pairs corresponding to $\hat{L}_1, \ldots, \hat{L}_{\hat{r}}$.

The intuition behind this approach is as follows: suppose among in total $p_0$ pairs of the components of $x_t$ there are $r$ connected pairs only. Arrange the true maximum cross correlations in the descending order: $L_1 \geq \cdots \geq L_{p_0}$. Then $L_r > 0$ and $L_{r+1} = 0$, and the ratio $L_j/L_{j+1}$ takes value $\infty$ for $j = r$. This motivates the estimator $\hat{r}$ defined in (2.13) in which we exclude some minimum $\hat{L}_r$ in the search for $\hat{r}$ as $c_0 \in (0, 1)$. This is to avoid the fluctuations due to the ratios of extreme small values such as “0/0”. This causes little loss in information as, for example, $0.75 p_0$ connected pairs would likely group most, if not all, component series together. The similar idea has been used in defining the factor dimensions in Lam and Yao (2012) and Chang et al. (2013a).

To state the asymptotic property of the above approach, we may use a graph representation. Let the graph contain $p$ vertexes $V = \{1, \ldots, p\}$, representing $p$ component series of $\bar{z}_t$. Define an edge connecting vertexes $i$ and $j$ if $H_0$ is rejected by the above ratio method for the pair $(\bar{z}_{i,t}, \bar{z}_{j,t})$. Let $\hat{E}_n$ be the set consisting all those edges. Let $V = \{1, \ldots, p\}$ represent the $p$ component series of $z_t = \Gamma'_y y_t$ defined in (2.8), and write $z_t = (z_{1,t}, \ldots, z_{p,t})'$. Define

$$
E = \{(i, j) : \max_{|h| \leq m} |\text{Corr}(z_{i,t+h}, z_{j,t})| > 0, 1 \leq i < j \leq p\}.
$$

Each $(i, j) \in E$ can be reviewed as an edge. We expect that the graph $(\hat{V}, \hat{E}_n)$ is a consistent estimate for the graph $(V, E)$; see Proposition 2 below. However for the technical reasons, we
need to modify (2.13) as follows:

\[ \hat{r} = \arg \max_{1 \leq j < p_0} \frac{\hat{L}_j + \delta_n}{(\hat{L}_{j+1} + \delta_n)}, \]  

(2.14)

where \( \delta_n > 0 \) is a small constant. Assume

\[ \min_{(i,j) \in E} \max_{|h| \leq m} |\text{Corr}(z_{i,t+h}, z_{j,t})| \geq \epsilon_n \]  

(2.15)

for some \( \epsilon_n \) such that \( n\epsilon_n^2 \to \infty \), and define

\[ \varpi_n = \min_{1 \leq i < j \leq p_0} \min_{\lambda \in \sigma(W_{x,i})} \min_{\mu \in \sigma(W_{x,j})} |\lambda - \mu|, \]  

(2.16)

where \( W_{x,i} \) is defined in (2.9), \( \sigma(W_{x,i}) \) denotes the set consisting of all the eigenvalues of \( W_{x,i} \). Hence \( \epsilon_n \) denotes the weakest signal to be identified in \( E \), and \( \varpi_n \) is the minimum difference between the eigenvalues from the different diagonal blocks in \( W_x \). Both \( \epsilon_n \) and \( \varpi_n \) may decay to 0 when dimension \( p \) and sample size \( n \) tend to \( \infty \). Arrange the true maximum cross correlations of \( z_t \) in the descending order \( L_1 \geq \cdots \geq L_{p_0} \) and define

\[ \chi_n = \max_{1 \leq j < r-1} \frac{L_j}{L_{j+1}}, \]  

where \( r = |E| \). Now we state the consistency in Proposition 2 below, which requires \( \varpi_n > 0 \) [see Proposition 1(ii) above]. This condition excludes some trivial cases. Recall that \( \hat{S} \) is the consistent estimator for \( W_y \) used in Step 1 in Section 2.1 above. Let

\[ \hat{\Sigma}_y(h) = \frac{1}{n} \sum_{t=1}^{n-h} (y_{t+h} - \bar{y})(y_t - \bar{y})' \quad \text{and} \quad \bar{y} = \frac{1}{n} \sum_{t=1}^{n} y_t. \]  

(2.17)

The proof of Proposition 2 is similar to that of Theorem 2.4 of Chang et al. (2013a), and is therefore omitted.

**Proposition 2.** Let \( \chi_n \delta_n = o(\epsilon_n) \) and \( \varpi_n^{-1} \|\hat{S} - W_y\|_2 = o_p(\delta_n) \). Let the singular values of \( \hat{\Sigma}_y(h) \) be uniformly bounded away from infinity for all \( |h| \leq m \). Then for \( \hat{r} \) defined in (2.14), it holds that

\[ P(\hat{E}_n = E) \to 1. \]

**Remark 2.** (i) The inserting of \( \delta_n \) in the definition of \( \hat{r} \) in (2.14) is to avoid the undetermined “0/0” cases. In practice, we use \( \hat{r} \) defined by (2.13) instead, but with the search restricted to \( 1 \leq j \leq c_0 p_0 \), as \( \delta_n \) subscribed in Proposition 2 is unknown. The simulation results reported in Section 4.2 indicates that (2.13) works reasonably well. See also Lam and Yao (2012) and Chang et al. (2013a).

(ii) The uniform boundedness for the singular values of \( \hat{\Sigma}_y(h) \) was introduced to simplify the presentation. The condition can be weakened. In fact it can be replaced by the condition

\[ \varpi_n^{-1} \nu_n \|\hat{S} - W_y\|_2 = o_p(\delta_n), \]  

where \( \nu_n \) is determined by

\[ \max_{|h| \leq m} \left\| \frac{1}{n} \sum_{t=1}^{n-h} (y_{t+h} - \bar{y})(y_t - \bar{y})' \right\|_2 = O_p(\nu_n). \]
(iii) The finite sample performance can be improved by prewhitening each component series \( \hat{z}_{i,t} \) first, \( i = 1, \ldots, p \). Then the asymptotic variance of \( \hat{\rho}_{i,j}(h) \) defined in (2.11) is \( 1/n \) as long as \( \text{Corr}(z_{i,t+h}, z_{j,t}) = 0 \), see Corollary 7.3.1 of Brockwell and Davis (1996). This makes the maximum cross correlations for different pairs more comparable. Note that two weakly stationary time series are correlated if and only if their prewhitened series are correlated.

### 2.2.3 FDR based on multiple tests

Alternatively we can identify the connected pair components of \( \hat{z}_t \) by a false discovery rate (FDR) procedure built on the multiple tests for cross correlations of each pair series.

In the same spirit of Remark 2(iii) above, we first prewhiten each component series of \( \hat{z}_t \) separately, and then look into the cross correlations of the prewhitened series which are white noise. Thus we only need to test hypothesis (2.10) for two white noise series.

To fix the idea, let \( \xi_t \) and \( \eta_t \) denote two white noise series. Let \( \rho(k) = \text{Corr}(\xi_{t+k}, \eta_t) \), and

\[
\hat{\rho}(k) = \frac{\sum_{i=1}^{n-k}(\xi_{t+k+i} - \bar{\xi})(\eta_{t+i} - \bar{\eta})}{\sqrt{\sum_{i=1}^{n}(\xi_t - \bar{\xi})^2 \sum_{i=1}^{n}(\eta_t - \bar{\eta})^2}}
\]

where \( \bar{\xi} = \frac{1}{n} \sum_{i=1}^{n} \xi_t \), \( \bar{\eta} = \frac{1}{n} \sum_{i=1}^{n} \eta_t \). By Theorem 7.3.1 of Brockwell and Davis (1996), \( \hat{\rho}(k) \sim N(0, n^{-1}) \) asymptotically, and \( \hat{\rho}(k), \hat{\rho}(j) \), for any \( k \neq j \), are asymptotically independent as \( n \to \infty \), provided that \( \rho(i) = 0 \) for all \( i \). Hence the \( P \)-value for testing a simple hypothesis \( \rho(k) = 0 \) based on statistic \( \hat{\rho}(k) \) is approximately equal to

\[
p_k = 2\Phi(-\sqrt{n}|\hat{\rho}(k)|),
\]

where \( \Phi(\cdot) \) denotes the distribution function of \( N(0, 1) \). Let \( p_{(1)} \leq \cdots \leq p_{(2m+1)} \) be the order statistics of \( \{p_k, k = 0, \pm 1, \ldots, \pm m\} \). As these \( P \)-values are approximately independent for large \( n \), a multiple test at the significant level \( \alpha \in (0, 1) \) rejects \( H_0 \), defined in (2.10), if

\[
p_{(j)} \leq j\alpha/(2m + 1) \quad \text{for at least one } 1 \leq j \leq 2m + 1.
\]

See Simes (1986) for details. In fact Sarkar and Chang (1997) showed that it is still a valid test at the level \( \alpha \) if \( \hat{\rho}(k) \), for different \( k \), are positive-dependent. Hence the \( P \)-value for this multiple test for the null hypothesis \( H_0 \) is

\[
P = \min_{1 \leq j \leq 2m+1} p_{(j)} (2m + 1)/j.
\]

The prewhitening is necessary in conducting the multiple test above, as otherwise \( \hat{\rho}(k) \) and \( \hat{\rho}(j) \) \( (k \neq j) \) are not asymptotically independent.

We can calculate the \( P \)-value for testing \( H_0 \) in (2.10) for each pair of the components of \( \hat{z}_t \), resulting in the total \( p_0 = p(p - 1)/2 \) \( P \)-values. Arranging those \( P \)-values in ascending order: \( P_{(1)} \leq \cdots \leq P_{(p_0)} \). For a given small \( \beta \in (0, 1) \), let

\[
d = \max\{k : 1 \leq k \leq p_0, P_{(k)} \leq k\beta/p_0\}.
\]
Then the FDR procedure with the error rate controlled under $\beta$ rejects the hypothesis $H_0$ for the $d$ pairs of the components of $\hat{z}_t$ corresponding to the $P$-values $P_{(1)}, \ldots, P_{(d)}$, i.e. those $d$ pairs of components are connected. Since the $P$-values $P_j$'s are no longer independent, the $\beta$ in (2.20) no longer admits the standard FDR interpretation. Nevertheless this procedure ranks the pair series of $\hat{z}_t$ accordingly to the strength of the cross correlations, which matters most as far as the dimension-reduction is concerned.

3 Theoretical properties

To gain more appreciation of the new methodology, we now investigate the asymptotic properties of the estimator $\hat{\Gamma}_y$ derived in Step 1 of the proposed method in Section 2.1. More precisely we will show that there exists a permutation transformation which permutes the column vectors of $\hat{\Gamma}_y$, and the resulting new orthogonal matrix, denoted as $\hat{A} = (\hat{A}_1, \ldots, \hat{A}_q)$, is an adequate estimator for the transformation matrix $A$ in (2.1) in the sense that $M(\hat{A}_j)$ is a consistent estimator for $M(A_j)$, $j = 1, \ldots, q$. In this section, we treat this permutation transformation as an ‘oracle’. In practice it is identified by the methods presented in Section 2.2 above. Our goal here is to show that $\hat{\Gamma}_y$ is a valid estimator for $A$ up to a column permutation. We establish the consistency under three different asymptotic modes: (i) the dimension $p$ is fixed, (ii) $p = o(n^c)$, and (iii) $\log p = o(n^c)$, as the sample size $n \to \infty$, where $c > 0$ is a constant. The convergence rates derived reflect the orders of the asymptotic estimation errors when, respectively, (i) $p$ is much smaller than $n$, (ii) $p$ and $n$ are of comparable sizes, and (iii) $p$ is much greater than $n$.

To measure the errors in estimating $M(A_j)$, we adopt a metric on the Grassmann manifold of $r$-dimensional subspaces of $\mathbb{R}^p$: for two $p \times r$ half orthogonal matrices $H_1$ and $H_2$ satisfying the condition $H_1' H_1 = H_2' H_2 = I_r$, the distance between $M(H_1)$ and $M(H_2)$ is defined as

$$D(M(H_1), M(H_2)) = \sqrt{1 - \frac{1}{r} \text{tr}(H_1' H_1 H_2' H_2)}.$$  

(3.1)

It can be shown that $D(M(H_1), M(H_2)) \in [0, 1]$. Furthermore it is equal to 0 if and only if $M(H_1) = M(H_2)$, and to 1 if and only if $M(H_1)$ and $M(H_2)$ are orthogonal with each other.

We always assume that the weakly stationary process $y_t$ is $\alpha$-mixing, i.e. its mixing coefficients $\alpha_k \to 0$ as $k \to \infty$, where

$$\alpha_k = \sup_i \sup_{A \in \mathcal{F}_{-\infty}^i, B \in \mathcal{F}_{i+k}^\infty} |P(A \cap B) - P(A)P(B)|,$$

(3.2)

and $\mathcal{F}_i^j$ is the $\sigma$-field generated by $\{y_t : i \leq t \leq j\}$ $(i \leq j)$. In sequel, we denote by $\sigma_{i,j}^{(k)}$ the $(i,j)$-th element of $\Sigma_y(k)$ for each $i, j = 1, \ldots, p$ and $k = 1, \ldots, k_0$. 


3.1 Asymptotics when \( n \to \infty \) and \( p \) fixed

When the dimension \( p \) is fixed, we estimate \( W_y \) defined in (2.5) by the plug-in estimator

\[
\hat{S} = I_p + \sum_{k=1}^{k_0} \hat{\Sigma}_y(k) \hat{\Sigma}_y(k)',
\]

where \( \hat{\Sigma}_y(k) \) is defined in (2.17). We show that the standard root-\( n \) convergence rate prevails as now \( p \) is fixed. We introduce some regularity conditions first.

**Condition 1.** For some constant \( \gamma > 2 \), \( E(|y_{i,t} - \mu_i|^{2\gamma}) \) is uniformly bounded away from infinity for any \( i = 1, \ldots, p \) and \( t \geq 1 \), where \( y_{i,t} \) and \( \mu_i \) are, respectively, the \( i \)-th component of \( y_t \) and \( \mu \equiv E(y_t) \).

**Condition 2.** The mixing coefficients \( \alpha_k \) defined in (3.2) satisfy the condition \( \sum_{k=1}^{\infty} \alpha_k^{1-2/\gamma} < \infty \), where \( \gamma > 0 \) is given in Condition 1.

**Theorem 1.** Let Conditions 1 and 2 hold, \( \varpi_n \) defined in (2.16) be positive, and \( p \) be fixed. Then

\[
\max_{1 \leq j \leq q} D(\mathcal{M}(\hat{A}_j), \mathcal{M}(A_j)) = O_p(n^{-1/2}),
\]

where the columns of matrix \( \hat{\mathbf{A}} = (\hat{A}_1, \ldots, \hat{A}_q) \) are a permutation of the columns of \( \hat{\Gamma}_y \).

3.2 Asymptotics when \( n \to \infty \) and \( p = o(n^\epsilon) \)

In the contemporary statistics dealing with large data, conventional wisdom assumes that \( \|\hat{S} - W_y\|_F = O_p(pm^{-1/2}) \) when \( p \), i.e. the dimension of time series, diverges together with sample size \( n \). Therefore it is necessary that \( pm^{-1/2} \to 0 \) in order to retain the consistency (but with a slower convergence rate than root-\( n \)). This means that the dimension \( p \) can only be as large as \( p = o(n^{1/2}) \) if we do not entertain any additional assumptions on the underlying structure. In order to deal with larger \( p \), we impose in Condition 3 below the sparsity on the transformation matrix \( A \) in (2.1).

**Condition 3.** Denote by \( a_{i,j} \) the \((i,j)\)-th element of \( A \). For some constant \( \iota \in [0,1) \),

\[
\max_{1 \leq j \leq p} \sum_{i=1}^{p} |a_{i,j}|^\iota \leq s_1 \quad \text{and} \quad \max_{1 \leq i \leq p} \sum_{j=1}^{p} |a_{i,j}|^\iota \leq s_2,
\]

where \( s_1 \) and \( s_2 \) are positive constants which may diverge together with \( p \).

When \( p \) is fixed, Condition 3 holds for \( s_1 = s_2 = p \) and any \( \iota \in [0,1) \), as \( A \) is an orthogonal matrix. For large \( p \), \( s_1 \) and \( s_2 \) control the degree of the sparsity of the columns and the rows of \( A \) respectively. A small \( s_1 \) entails that each component of the latent process \( x_t \) only contributes to a small fraction of the components of \( y_t \). A small \( s_2 \) entails that each component of \( y_t \) is a linear combination of a small number of the components of \( x_t \). The sparsity of \( A \) is also controlled by
constant $\iota$: the smaller $\iota$ is, more sparse $\mathbf{A}$ is. We will show that the stronger sparsity leads to the faster convergence for our estimator; see Remark 3(ii) below.

When $p$ diverges together with $n$, the sample auto-covariance matrix $\hat{\mathbf{\Sigma}}_y(k) \equiv (\hat{\sigma}_{i,j}^{(k)})_{i,j=1,\ldots,p}$, given in (2.17), is no longer a consistent estimator for $\mathbf{\Sigma}_y(k)$. We employ a threshold estimator instead:

$$ T_u(\hat{\mathbf{\Sigma}}_y(k)) = (\hat{\sigma}_{i,j}^{(k)})_{i,j=1,\ldots,p}, $$

where $\mathbb{I}(\cdot)$ is the indicator function, $u =Mp^{2/}(n^{1/2})$ is the threshold level, and $M > 0$ is a constant. This threshold value is chosen as $\max_{i,j}|\hat{\sigma}_{i,j}^{(k)} - \sigma_{i,j}^{(k)}| = O_p(p^{2/}n^{1/2})$, see Lemma 5 in Section 7 below. Consequently, we define now

$$ \hat{\mathbf{S}} = \hat{\mathbf{W}}_y^{(\text{thre})} = \mathbf{I}_p + \sum_{k=1}^{k_0} T_u(\hat{\mathbf{\Sigma}}_y(k))T_u(\hat{\mathbf{\Sigma}}_y(k))'. $$

Lemma 8 in Section 7 shows that $\hat{\mathbf{W}}_y^{(\text{thre})}$ is a consistent estimator for $\mathbf{W}_y$, which requires a stronger version of Conditions 1 and 2 as now $p$ diverges together with $n$.

**Condition 4.** For some positive constants $l > 2$ and $\tau > 0$,

$$ \sup_{t} \sup_{i} P(|y_{i,t} - \mu_i| > x) = O(x^{-2(l+\tau)}) \text{ as } x \to \infty. $$

**Condition 5.** The mixing coefficients given in (3.2) satisfy the condition

$$ \alpha_k = O(k^{-l(l+\tau)/(2\tau)}) \text{ as } k \to \infty, $$

where $l$ and $\tau$ are given in Condition 4.

Conditions 4 and 5 ensure the Fuk-Nagaev type inequalities for $\alpha$-mixing processes, see Rio (2000). Condition 4 is implied by the moment condition $\sup_{t} \sup_{i} E\{|y_{i,t} - \mu_i|^{2(l+\tau)}\} < \infty$. Under Condition 5, Condition 2 holds with $\gamma = l + \tau$. To state the asymptotic results, we introduce more notations. Let

$$ \rho_j = \min_{1 \leq i \leq q} \min_{\lambda \in \sigma(\mathbf{W}_{x,j})_{i,j} \neq \mu \in \sigma(\mathbf{W}_{x,j})} |\lambda - \mu|, \quad j = 1, \ldots, q. $$

(3.6)

Put

$$ \delta = s_1s_2 \max_{1 \leq j \leq q} \rho_j \text{ and } \kappa = \max_{1 \leq k \leq k_0} \|\mathbf{\Sigma}_x(k)\|_2. $$

(3.7)

Now we let $\hat{\mathbf{S}} = \hat{\mathbf{W}}_y^{(\text{thre})}$ in Step 1 in our estimation method. Then we have the following theorem.

**Theorem 2.** Let Conditions 3, 4 and 5 hold, $\rho_j > 0$ for $j = 1, \ldots, q$, and $p = o(n^{1/4})$ for $l$ given in Condition 4. Then there exists an $\hat{\mathbf{A}} = (\hat{\mathbf{A}}_1, \ldots, \hat{\mathbf{A}}_q)$ of which the columns are a permutation of the columns of $\hat{\mathbf{Y}}_y$, such that

$$ \max_{1 \leq j \leq p} \rho_j D(\mathcal{M}(\hat{\mathbf{A}}_j), \mathcal{M}(\mathbf{A}_j)) = O_p\{\kappa (p^{4/}n^{-1})^{(1-\iota)/2} \delta + (p^{4/}n^{-1})^{1-\iota} \delta^2\}. $$

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Remark 3. (i) Theorem 2 presents the uniform convergence rate for \( \rho_j D(\mathcal{M}(\hat{A}_j), \mathcal{M}(A_j)) \). Since \( \rho_j \) measures the minimum difference between the eigenvalue of \( \mathcal{M}(A_j) \) and those of the other blocks [see (3.6)], it is intuitively clear that the smaller this difference is, more difficult the estimation for \( \mathcal{M}(A_j) \) is. See a similar phenomenon in estimating the principal subspaces with independent observations in Vu and Lei (2013).

(ii) As \( \Sigma_y(k) = A \Sigma_x(k) A' \), the maximum block size \( S_{\text{max}} = \max_{1 \leq j \leq q} p_j \) and the sparsity of \( A \) determine the sparsity of \( \Sigma_y(k) \). As shown in Lemma 6 in Section 7, the sparsity of \( \Sigma_y(k) \) can be evaluated by \( \delta \) defined in (3.7). A small value of \( S_{\text{max}} \) represents a high degree of sparsity for \( \Sigma_x(k) \) and, therefore, also for \( \Sigma_y(k) \), while the sparsity of \( A \) is reflected by \( \iota, s_1 \), and \( s_2 \); see Condition 3 and the comments immediately below Condition 3. The convergence rates presented in Theorem 2 contain factors \( \delta \) or \( \delta^2 \). Hence more sparse \( \Sigma_y(k) \) is (i.e. the smaller \( \delta \) is), and the faster the convergence is.

(iii) With the sparsity imposed in Condition 3, the dimension of time series can be as large as \( p = o(n^{l/4}) \), where \( l > 2 \) is determined by the tail probabilities described in Condition 4.

3.3 Asymptotics when \( n \to \infty \) and \( \log p = o(n^c) \)

To handle the ultra high-dimensional cases when the dimension of \( y_t \) grows at an exponential rate of the sample size, we need even stronger conditions (than Conditions 4 and 5) on the decays of the tail probabilities of \( y_t \) and the mixing coefficients \( \alpha_k \) defined in (3.2). See below.

**Condition 6.** It holds for any \( x > 0 \) and \( \|v\|_2 = 1 \) that

\[
\sup_{t} P\{|v^t(y_t - \mu)| > x\} \leq K_1 \exp(-K_2 x^{r_1}),
\]

where \( K_1, K_2 > 0 \), and \( r_1 \in (0, 2] \) are constants.

**Condition 7.** It holds for all \( k \geq 1 \) that \( \alpha_k \leq \exp(-K_3 k^{r_2}) \), where \( K_3 > 0 \) and \( r_2 \in (0, 1] \) are some constants.

Condition 6 requires the tail probabilities of linear combinations of \( y_t \) decay exponentially fast. When \( r_1 = 2 \), \( y_t \) is sub-Gaussian. It is also intuitively clear that the large \( r_1 \) and/or \( r_2 \) would only make Conditions 6 and/or 7 stronger. The restrictions \( r_1 \leq 2 \) and \( r_2 \leq 1 \) are introduced only for the presentation convenience, as Theorem 3 below applies to the ultra high-dimensional cases with

\[
\log p = o(n^{\varrho/(2-\varrho)}), \quad \text{where} \quad \varrho = 1/(2r_1^{-1} + r_2^{-1}). \tag{3.8}
\]

We still use \( \hat{S} = \hat{W}_y^{(\text{thre})} \) defined in (3.5) in Step 1 in our procedure, as in section 3.2 above. But now the threshold value should be set as \( u = M(n^{-1} \log p)^{1/2} \) in the transacted sample covariance matrix \( T_u(\hat{\Sigma}_y(k)) \) in (3.4), as \( \max_{i,j} |\hat{\sigma}_{ij}^{(k)} - \sigma_{ij}^{(k)}| = O_p\{n^{-1} \log p\}^{1/2} \} \) when \( p \) is specified by (3.8); see Lemma 9 in Section 7. Recall that \( \delta \) and \( \kappa \) are defined in (3.7). Now we are ready to state the asymptotic results.
Theorem 3. Let Conditions 3, 6 and 7 hold, and \( p \) be specified in (3.8). Let \( \rho_j \) defined in (3.6) be positive for all \( 1 \leq j \leq q \). Then there exists an \( \hat{A} = (\hat{A}_1, \ldots, \hat{A}_q) \) of which the columns are a permutation of the columns of \( \hat{\Gamma}_y \), such that

\[
\max_{1 \leq j \leq q} \rho_j D(\mathcal{M}(\hat{A}_j), \mathcal{M}(A_j)) = O_p\left\{ \kappa(n^{-1} \log p)^{1/2} + (n^{-1} \log p)^{1/2} \delta^2 \right\}.
\]

4 Numerical Properties

We illustrate the proposed methodology with both real and simulated data. The segmentation is only possible if such a latent structure exists, as assumed in (2.1). Two questions arise immediately: (i) Is such an assumption of substantial practical relevance? (ii) What would the proposed method lead to if (2.1) does not hold? To answer these questions, we apply the proposed method to several real data sets arising from different fields. We then report some simulation studies which illustrate the finite sample properties of the proposed method.

Unless specifically stated, we always standardize the data using the sample covariance matrix, i.e. to replace \( y_t \) by \( \hat{\Sigma}_y(0)^{-1/2}y_t \); see (2.2) and (2.17). We always prewhiten each transformed component series before applying the permutation methods described in Section 2.2. The prewhitening is carried out by fitting each series an AR model with the order between 0 and 5 determined by AIC. The resulting residual series is taken as a prewhitened series. We set the upper bound for the AR-order at 5 to avoid over-whitening with finite samples. We always set \( c_0 = 0.75 \) in (2.13) and \( k_0 = 5 \) in (3.3) unless specified otherwise.

4.1 Illustration with real data

For each multiple time series \( y_t \), we apply the proposed method to seek for a segmentation transformation \( \hat{x}_t = \hat{B}y_t \), where \( \hat{B} = \hat{\Gamma}_y^T \hat{\Sigma}_y(0)^{-1/2} \) is the cumulative transformation matrix resulted from first the standardization transformation for \( y_t \) [see (2.2)], and the segmentation transformation in Step 1 of the proposed method described in Section 2.1. We report \( \hat{B} \) for each real data set below, except Example 4 for which \( \hat{B} \) is too big (with \( p = 25 \)).

To show the advantages of the proposed segmentation method, we also conduct post-sample forecasting and compare the performance with that based on direct VAR modelings. To this end, we use the R-function \textit{VAR} in the R-package \textit{vars} to fit VAR models with orders determined by AIC. We also report the results from the restricted VAR model (RVAR) obtained by setting insignificant coefficients to 0 in a fitted VAR model, using the R-function \textit{restrict} in the package \textit{vars}.

Example 1. Our first example concerns the monthly temperatures (in Celsius) in January 1954 – December 1998 in 7 cities (i.e. Nanjing, Dongtai, Huoshan, Hefei, Shanghai, Anqing and Hangzhou) in Eastern China; see Fig.1. Fig.2 are the plots of the cross correlations of those 7 temperature time series. Both the autocorrelation of each component series and the cross corre-
lation between any two component series are dominated by the annual temperature fluctuation; showing the strong periodicity with the period 12. For this data set, \( n = 540 \) and \( p = 7 \).

We apply the proposed method to this 7-dimensional time series, denoted as \( \mathbf{y}_t \). The resulting transformed series \( \hat{\mathbf{x}}_t = \hat{\mathbf{B}} \mathbf{y}_t \) are plotted in Fig.3. Their cross correlation coefficients are plotted in Fig.4. The cumulative transformation matrix is

\[
\hat{\mathbf{B}} = \begin{pmatrix}
0.244 & -0.066 & 0.019 & -0.050 & -0.313 & -0.154 & 0.200 \\
-0.703 & 0.324 & -0.617 & 0.189 & 0.633 & 0.499 & -0.323 \\
0.375 & 1.544 & -1.615 & 0.170 & -2.266 & 0.126 & 1.596 \\
3.025 & -1.381 & -0.787 & -1.691 & -0.212 & 1.188 & -0.165 \\
-0.197 & -1.820 & -1.416 & 3.269 & 0.301 & -1.438 & 1.299 \\
-0.584 & -0.354 & 0.847 & -1.262 & -0.218 & -0.151 & 1.831 \\
1.869 & -0.742 & 0.034 & 0.501 & 0.492 & -2.533 & 0.339
\end{pmatrix}.
\]

It is easy to see from Fig.4 that the first two transformed component series are significantly correlated both concurrently and serially, and there are also small but significant correlations in the \((3, 2)\)-th panel in the figure; indicating the correlations between the 2nd and 3rd component series. Apart from these, there is little significant cross correlation among all the other pairs of component series. This visual observation suggests to segment the 7 transformed series, plotted in Fig.3 into 5 uncorrelated groups: \{1, 2, 3\}, \{4\}, \{5\}, \{6\} and \{7\}. We used the plug-in sample cross covariance matrices to estimate \( \mathbf{W}_y \) in (2.5) with \( k_0 = 5 \). The profile of the grouping is unchanged for \( 2 \leq k_0 \leq 36 \).

The exactly the same grouping is obtained by the permutation based on the maximum cross correlation method with \( m \) in (2.10) between 2 and 30 (Section 2.2.2), or by the permutation based on FDR with \( 2 \leq m \leq 30 \) and \( \beta \) in (2.20) between 1\% and 0.001\% (Section 2.2.3).

Forecasting the original time series \( \mathbf{y}_t \) can be carried out in two steps: First we forecast the components of \( \hat{\mathbf{x}}_t \) using 5 models according to the segmentation, i.e. one VAR for the first three components, and a univariate AR for each of the last four components. Then the forecasted values for \( \mathbf{y}_t \) are obtained via the transformation \( \mathbf{y}_t = \hat{\mathbf{B}}^{-1} \hat{\mathbf{x}}_t \). For each of the last 24 values in this data set (i.e. the monthly temperatures in 1997-1998), we use the data up to the previous month to fit three forecasting models, i.e. the model based on the segmentation (which is a collection of 5 AR models for the 5 segmented subseries of \( \hat{\mathbf{x}}_t \)), the VAR model, and the restricted VAR (RVAR) model. We difference the original data at lag 12 before fitting them directly with VAR and RVAR models, to remove the seasonal components. For fitting the segmented series, we only difference the first two component series also at lag 12; see Fig.3. The one-step-ahead forecasts can be obtained directly from the fitted AR models. The two-step-ahead forecasts are obtained based on the plug-in method, i.e. using the one-step-ahead forecasted values as true values. We calculate the mean squared errors (MSE) for both one-step-ahead and two-step-ahead forecasting for each
of the 7 cities. For example, the MSE for forecasting the \( i \)-th component \( y_t \) is

\[
\frac{1}{d} \sum_{h=1}^{d} (\hat{y}_{i,n+h} - y_{i,n+h})^2,
\]

where \( \hat{y}_{i,n+h} \) denotes the one-step-ahead or the two-step-ahead forecast for \( y_{i,n+h} \). (For this example, \( d = 24 \).) The mean and the standard deviations of those MSE over the 7 cities are listed in Table 1. Both the mean and the standard deviation based on the proposed segmentation are much smaller than those based on the direct VAR models or the restricted VAR models.

Table 1: Means and standard deviations (in subscripted bracket) of the MSE for one-step-ahead and two-step-ahead forecasts for the monthly temperatures over the 7 cities in 1997-1998.

<table>
<thead>
<tr>
<th></th>
<th>One-step MSE</th>
<th>Two-step MSE</th>
</tr>
</thead>
<tbody>
<tr>
<td>VAR</td>
<td>1.669(2.355)</td>
<td>1.815(2.372)</td>
</tr>
<tr>
<td>RVAR</td>
<td>1.677(2.324)</td>
<td>1.829(2.398)</td>
</tr>
<tr>
<td>Segmentation</td>
<td>1.394(1.896)</td>
<td>1.647(2.017)</td>
</tr>
</tbody>
</table>

**Example 2.** Now we consider the 8 monthly US Industrial Production indices in January 1947 – December 1993 published by the US Federal Reserve. The 8 indices concerned are the total index, manufacturing index, durable manufacturing, nondurable manufacturing, mining, utilities, products and materials. Since those index series exhibit clearly nonstationary trends, we difference each series first. The 8 differenced series are plotted in Fig.5 with their cross correlogram displayed in Fig.6. It is clear from Fig.6 that those 8 differenced series cannot be segmented into smaller subgroups. We apply the proposed method to the 8-dimensional differenced indices. The resulting transformation matrix is

\[
\hat{B} = \begin{pmatrix}
5.012 & -1.154 & -0.472 & -0.880 & -0.082 & -0.247 & -2.69 & -1.463 \\
10.391 & 8.022 & -3.981 & -3.142 & 0.186 & 0.019 & -6.949 & -4.203 \\
-6.247 & 11.879 & -4.885 & -4.044 & 0.289 & -0.011 & 2.557 & 0.243 \\
1.162 & -6.219 & 3.163 & 1.725 & 0.074 & -0.823 & 0.646 & -0.010 \\
6.172 & -4.116 & 2.958 & 1.887 & 0.010 & 0.111 & -2.542 & -3.961 \\
0.868 & 1.023 & -2.946 & -4.615 & -0.271 & -0.354 & 3.972 & 1.902 \\
3.455 & -2.744 & 5.557 & 3.165 & 0.753 & 0.725 & -2.331 & -1.777 \\
0.902 & -2.933 & -1.750 & -0.123 & 0.191 & -0.265 & 3.759 & 0.987
\end{pmatrix}.
\]

The transformed series \( \hat{y}_t = \hat{B}y_t \), where \( y_t \) denotes the 8 differenced indices, are plotted in Fig.7 with its cross correlogram presented in Fig.8. A visual observation of Fig.8 would suggest no noticeable cross correlations in all the panels off the main-diagonal. But close examination of those off-diagonal panels reveals small but significant correlations in the panels at the positions (1, 2), (1, 3), (3, 1) and (8, 4). This suggests a segmentation with the 5 groups: \{1, 2, 3\}, \{4, 8\}, \{5\}, \{6\}, and \{7\}. This segmentation is also confirmed by the permutation based on FDR with \( m = 5 \) and
\( \beta = 0.005. \) However with \( m = 5 \) and \( \beta \in [10^{-6}, 0.001], \) or \( m = 20 \) and \( \beta \in [10^{-6}, 0.01], \) the permutation based on FDR leads a segmentation of 7 groups with \( \{1, 3\} \) as the only group containing more than one members. The permutation based on maximum cross correlation method, with \( 1 \leq m \leq 20 \) in (2.10), also entails this segmentation of the 7 groups. Looking at the correlogram in Fig.8, there is no need to use large values for \( m \). Since those significant cross correlations are so small, we accept both the segmentations with the 5 or the 7 groups as viable options for initial dimension reduction in analysing the original 8-dimensional time series.

We carry out the forecast comparison in the same manner as in Example 1. Namely we forecast the monthly indices in January 1992 – December 1993 based on the segmentation, direct VAR and RVAR methods. For the segmentation approach, we use the option with the 7 groups. This means that we ignore some small, though significant, cross correlations. We argue that those small correlations are of little practical value. The results are reported in Table 2. The proposed segmentation method outperforms the other two methods, though the restricted VAR represents a significant improvement over the direct VAR for this particular data set.


<table>
<thead>
<tr>
<th>Method</th>
<th>One-step MSE</th>
<th>Two-step MSE</th>
</tr>
</thead>
<tbody>
<tr>
<td>VAR</td>
<td>0.615(1.349)</td>
<td>1.168(2.129)</td>
</tr>
<tr>
<td>RVAR</td>
<td>0.606(1.293)</td>
<td>1.159(2.285)</td>
</tr>
<tr>
<td>Segmentation</td>
<td>0.588(1.341)</td>
<td>1.154(2.312)</td>
</tr>
</tbody>
</table>

**Example 3.** Fig.9 displays the weekly notified measles cases in 7 cities in England (i.e. London, Bristol, Liverpool, Manchester, Newcastle, Birmingham and Sheffield) in 1948-1965, before the advent of vaccination. All the 7 series show biennial cycles, which is a common feature in measles dynamics in the pre-vaccination period. This biennial cycling is the major driving force for the cross correlations among different component series displayed in Fig.10. The sample size is \( n = 937. \)

Applying the proposed transformation to the 7-dimensional measles series, the resulting series are plotted in Fig.11 with their cross correlogram displayed in Fig.12. The transformation matrix is

\[
\hat{B} = \begin{pmatrix}
-4.898e4 & 3.357e3 & -3.315e4 & -6.455e3 & 2.337e3 & 1.151e3 & -1.047e3 \\
7.328e4 & 2.850e4 & -9.569e6 & -2.189e3 & 1.842e3 & 1.457e3 & 1.067e3 \\
-5.780e5 & 5.420e3 & -5.247e3 & 5.878e4 & -2.674e3 & -1.238e3 & 6.280e3 \\
-1.766e3 & 3.654e3 & 3.066e3 & 2.492e3 & 2.780e3 & 8.571e4 & 2.356e3 \\
-1.466e3 & -7.337e4 & -5.896e3 & 3.663e3 & 6.633e3 & 3.472e3 & -4.668e3 \\
-7.620e4 & -3.338e3 & 1.471e3 & 2.099e3 & -1.318e2 & 4.259e3 & 6.581e4 \\
\end{pmatrix}
\]
Table 3: Segmentations determined by different numbers of connected pairs for the transformed measles series for the 7 cities in England.

<table>
<thead>
<tr>
<th>No. of connected pairs</th>
<th>No. of groups</th>
<th>Segmentation</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>6</td>
<td>{4, 5}, {1}, {2}, {3}, {6}, {7}</td>
</tr>
<tr>
<td>2</td>
<td>5</td>
<td>{1, 2}, {4, 5}, {3}, {6}, {7}</td>
</tr>
<tr>
<td>3</td>
<td>4</td>
<td>{1, 2, 3}, {4, 5}, {6}, {7}</td>
</tr>
<tr>
<td>4</td>
<td>3</td>
<td>{1, 2, 3, 7}, {4, 5}, {6}</td>
</tr>
<tr>
<td>5</td>
<td>2</td>
<td>{1, 2, 3, 6, 7}, {4, 5}</td>
</tr>
<tr>
<td>6</td>
<td>1</td>
<td>{1, ..., 7}</td>
</tr>
</tbody>
</table>

The entries in above matrix are written in the format ‘aek’ standing for $a \times 10^{-k}$. Since none of the transformed component series are white noise, the confidence bounds in Fig.12 could be misleading; see Remark 1 above. We apply prewhitening to each time series plotted in Fig.11 by fitting an AR model with the order determined by AIC and with the maximum order set at 5. Now all those 7 filtered time series behave like white noise. But there are still quite a few small but significant cross correlations here and there. Fig. 13(a) plots, in descending order, the maximum cross correlations $\hat{\rho}_{ij}$ for $1 \leq i < j \leq 7$ for those 7 transformed and prewhitened series; see (2.12). As $1.96/\sqrt{n} = 0.064$ now, one may argue that the segmentation assumption (2.1) does not hold for this example. Consequently the ratio estimator $\hat{r}$ defined in (2.13) does not make any sense now; see also Fig.13(b).

Nevertheless Fig.13(a) ranks the pairs of transformed component series according to the strength of the cross correlation. If we would only accept $r$ connected pairs, this leads to an approximate segmentation according to the rule set in Section 2.2.1. By doing this, we effectively ignore some small, though still statistically significant, cross correlations. Table 3 lists the different segmentations corresponding to the different values of $r$. It shows that the group \{4, 5\} is always present until all the 7 series merge together. Further it only takes 6 connected pairs, corresponding to the 6 largest points in Fig.13(a), to merge all the series together.

The forecasting comparison is conducted in the same manner as in the previous two examples. We adopt the segmentation with 4 groups: \{1, 2, 3\}, \{4, 5\}, \{6\} and \{7\}, i.e. we regard that only the three pairs, corresponding to the 3 maximum cross correlations in Fig.13(a), are connected. We forecast the notified measles cases in the last 14 weeks of the period for all the 7 cities. The results are reported in Table 4. Once again the forecasting based on this (approximate) segmentation is much more accurate than those based on the direct VAR and RVAR, although we have ignored quite a few small but significant cross correlations among the transformed series. Note that the large values of the standard deviation reported in Table 4 are due to the fact that the data from different cities are on different scales; see Fig.9.

**Example 4.** Now we consider the daily sales of a clothing brand in 25 provinces in China in 1 January 2008 – 9 December 2012 (i.e. $n = 1805$ and $p = 25$). The sales in the natural logarithm
Table 4: Means and standard deviations (in subscripted bracket) of the MSE for one-step-ahead and two-step-ahead forecasts for the reported measles cases in the last 14 weeks of 1965 in the 7 cities in England.

<table>
<thead>
<tr>
<th></th>
<th>One-step MSE</th>
<th>Two-step MSE</th>
</tr>
</thead>
<tbody>
<tr>
<td>VAR</td>
<td>503.408$^{(1124.213)}$</td>
<td>719.499$^{(2249.986)}$</td>
</tr>
<tr>
<td>RVAR</td>
<td>574.582$^{(1432.217)}$</td>
<td>846.141$^{(2462.019)}$</td>
</tr>
<tr>
<td>Segmentation</td>
<td>472.106$^{(1088.170)}$</td>
<td>654.843$^{(1807.502)}$</td>
</tr>
</tbody>
</table>

scale in 8 provinces (i.e. Beijing, Fujian, Guangdong, Guangxi, Hainan, Hebei, Henan and Hubei) are plotted in Fig.14. All those series exhibit peaks before the Spring Festival (i.e. the Chinese New Year, typically around February). The cross-correlogram of those 8 series in Fig.15 indicates strong periodic components with the period 7; corresponding to a regular sales pattern over 7 different weekdays.

By applying the proposed segmentation transformation and the permutation based on the maximum cross correlations with $m = 25$ in (2.12), the transformed $25 \times 1$ vector time series is divided into 24 group with only non-single-element group containing the 15th and the 16th transformed components. The same grouping is obtained for $m$ between 14 and 30. Note for this example, we should not use small $m$ as the autocorrelations of the original data decay slowly; see Fig.15. Fig.16 plots 8 out of 25 transformed series, including the the 15th and the 16th components. Their cross correlogram is presented in Fig.17. It is clear from the correlogram that significant cross correlations only occur between the 15th and the 16th component series.

To compare the forecasting performance, we calculate one-step-ahead and two-step-ahead forecasts for each of the daily log-sales in the last two weeks of the period. Since the dimension $p = 25$ is large, we also include the forecasts based on the fitted univariate AR models for each of the 25 original sales series. By doing this we ignore the cross correlations among the sales from different provinces. Table 5 list the means and the standard deviations of the MSE across the 25 provinces. With $p = 25$, the fitted VAR(2) model, selected by AIC, contain $2 \times 25 \times 25 = 1250$ parameters, leading to poor post-sample forecasting. The restricted VAR(2) improves the forecasting a bit, but it still is significantly worse than the forecasting based on the approach of fitting a univariate AR to each of the original series directly. Since the proposed segmentation leads to 24 subseries, it also fits univariate AR models to 23 (out of 25) transformed series, fits a two-dimensional VAR model to the 15th and the 16th transformed series together. This new approach leads to much more accurate forecasts, as both the mean and the standard deviation of the MSE over the 25 provinces are much smaller than those based on the other three methods. The above comparison shows clearly that the cross correlations in the sales over different provinces are valuable information which can improve the forecasting for the future sales significantly. However a step to reduce the dimension by, for example, the proposed segmentation method, is necessary in order to make use this valuable information.
Table 5: Means and standard deviations (in subcripted bracket) of the MSE of one-step-ahead and two-step-ahead forecasts for the lag sales of a brand clothes in 26 November – 9 December 2012 in 25 provinces in China.

<table>
<thead>
<tr>
<th></th>
<th>One-step MSE</th>
<th>Two-step MSE</th>
</tr>
</thead>
<tbody>
<tr>
<td>univariate AR</td>
<td>0.208(1.255)</td>
<td>0.194(1.250)</td>
</tr>
<tr>
<td>VAR</td>
<td>0.295(2.830)</td>
<td>0.301(2.930)</td>
</tr>
<tr>
<td>RVAR</td>
<td>0.293(2.817)</td>
<td>0.296(2.962)</td>
</tr>
<tr>
<td>Segmentation</td>
<td>0.153(0.361)</td>
<td>0.163(0.341)</td>
</tr>
</tbody>
</table>

4.2 Simulation

In this section, we illustrate the finite sample properties of the proposed methodology using four simulated models. As the dimensions of $\mathcal{M}(A_j)$ and $\mathcal{M}(\hat{A}_j)$ are not necessarily the same, we extend the discrepancy measure (3.1) to a more general form below. Let $H_i$ be a $p \times r_i$ matrix with $\text{rank}(H_i) = r_i$, and $P_i = H_i(H_i'H_i)^{-1}H_i'$, $i = 1, 2$. Define

\[ \tilde{D}(\mathcal{M}(H_1), \mathcal{M}(H_2)) = \sqrt{1 - \frac{1}{\min(r_1, r_2)} \text{tr}(P_1P_2)}. \]

Then $\tilde{D} \in [0, 1]$. Furthermore, $\tilde{D}(\mathcal{M}(H_1), \mathcal{M}(H_2)) = 0$ if and only if either $\mathcal{M}(H_1) \subset \mathcal{M}(H_2)$ or $\mathcal{M}(H_2) \subset \mathcal{M}(H_1)$, and 1 if and only if $\mathcal{M}(H_1) \perp \mathcal{M}(H_2)$. When $r_1 = r_2 = r$ and $H_i'H_i = I_r$, $\tilde{D}(\mathcal{M}(H_1), \mathcal{M}(H_2)) = D(\mathcal{M}(H_1), \mathcal{M}(H_2))$ defined in (3.1).

Since the goal is to specify (via estimation) the $q$ linear spaces $\mathcal{M}(A_j)$, $j = 1, \ldots, q$, simultaneously, we first introduce the concept of a ‘correct’ specification. We call $\hat{A} = (\hat{A}_1, \ldots, \hat{A}_q)$ a correct specification for $A$ if (i) $\tilde{q} = q$, and (ii) $\text{rank}(\hat{A}_j) = \text{rank}(A_j)$ for $j = 1, \ldots, q$, after re-arranging the order of $\hat{A}_1, \ldots, \hat{A}_q$. (We still denote the rearranged order as $\hat{A}_1, \ldots, \hat{A}_q$ for the simplicity in notation.) When more than one $A_j$ have the same rank, we pair each those $A_j$ with the $\hat{A}_j$ for which

\[ \tilde{D}(\mathcal{M}(A_j), \mathcal{M}(\hat{A}_j)) = \min_{\text{rank}(\hat{A}_i) = \text{rank}(A_j)} \tilde{D}(\mathcal{M}(A_j), \mathcal{M}(\hat{A}_i)). \]

Note that a correct specification for $A$ implies a structurally correct segmentation for $x_t$, which will be abbreviated as ‘correct segmentation’ hereafter. For a correct segmentation, we report the estimation error defined as

\[ \tilde{D}(\hat{A}, A) = \frac{1}{q} \sum_{j=1}^q \tilde{D}(\mathcal{M}(A_j), \mathcal{M}(\hat{A}_j)). \quad (4.1) \]

As the estimated $\hat{A}$ is an orthogonal matrix for the ‘normalized’ model in which $\text{Var}(y_t) = \text{Var}(x_t) = I_p$. We should use $\hat{S}^{-1/2}A$ instead of $A$ in computing estimation error (4.1), where
\( \hat{S} \) is the adopted consistent estimator for \( \text{Var}(y_t) \). This can be seen as follows: let \( A^* = \Sigma_y(0)^{-1/2}A\Sigma_x(0)^{1/2} \equiv (A_1^*, \cdots, A_q^*) \), and \( \Sigma_y(0)^{-1/2}A = (H_1, \cdots, H_q) \). Since \( \Sigma_x(0)^{1/2} \) is a block-diagonal matrix, it holds that \( M(H_j) = M(A_j) \) for \( 1 \leq j \leq q \).

A partial dimension reduction may still be achieved by an incomplete segmentation which refers to the cases with \( \hat{q} < q \) and each identified subseries containing one or more original subseries of \( x_t \). We pay particular attention to the incomplete segmentation with \( \hat{q} = q - 1 \), i.e. \( (q - 2) \) subseries of \( x_t \) are correctly identified and the remaining two are put into one group.

In the simulation studies reported below, each setting is replicated 500 times with the elements of \( A \) drawn independently from \( U(-3, 3) \). For the permutation, only the maximum cross correlation method is used with \( m = 10 \log_{10}(n/p) \) unless specified otherwise, which is the default upper bound for lags used in the R-function \( \text{acf} \). We report the relative frequencies of both the correct segmentation and the incomplete segmentation. We also report the relative frequencies of the incomplete segmentation with \( \hat{q} = q - 1 \), which corresponds to the instances achieving good partial dimension reduction. We always use \( \hat{\varepsilon}_t^{(i)} \) denote independent and standard normal random variables.

**Example 5.** We consider model (2.1) with \( p = 6 \). The components of \( x_t \) are defined as follows.

\[
\begin{align*}
  x_{i,t} &= \eta_{t+i-1}^{(1)} & (i = 1, 2, 3), \\
  x_{i,t} &= \eta_{t+i-4}^{(2)} & (i = 4, 5) \quad \text{and} \\
  x_{6,t} &= \eta_{t}^{(3)},
\end{align*}
\]

where

\[
\begin{align*}
  \eta_{t}^{(1)} &= 0.5\eta_{t-1}^{(1)} + 0.3\eta_{t-2}^{(1)} + \varepsilon_{t}^{(1)} - 0.9\varepsilon_{t-1}^{(1)} + 0.3\varepsilon_{t-2}^{(1)} + 1.2\varepsilon_{t-3}^{(1)} + 1.3\varepsilon_{t-4}^{(1)}, \\
  \eta_{t}^{(2)} &= -0.4\eta_{t-1}^{(2)} + 0.5\eta_{t-2}^{(2)} + \varepsilon_{t}^{(2)} + \varepsilon_{t-1}^{(2)} - 0.8\varepsilon_{t-2}^{(2)} + 1.5\varepsilon_{t-3}^{(2)}, \\
  \eta_{t}^{(3)} &= 0.9\eta_{t-1}^{(3)} + \varepsilon_{t}^{(3)}.
\end{align*}
\]

Thus \( x_t \) consists of three independent subseries with, respectively, 3, 2 and 1 components. Simulation is conducted for the 10 different sample sizes between 100 and 3000. The proportions of correct segmentations and the incomplete segmentations are listed in Table 6. The boxplots for the estimation errors, defined as (4.1), in the instances of the correct segmentation are presented in Fig.20. As expected, the performance of the proposed method improves as the sample size increases. Nevertheless the method performs reasonably well with the sample size as small as 200 or 300. For example for \( n = 200 \), the proportion of correction segmentations is 56.4\% among the 500 replications. The further 25.8\% of instances end of the segmentation of \( \hat{q} = 2 \) subseries instead of \( q = 3 \).

In the above simulation (with 500 replications), we apply the permutation based on maximum cross correlation method to derive the grouping. With \( p = 6 \), such an analytic method is not necessary in practice. Fig.18 displays the cross correlogram of \( y_t \) for one instance. It shows that all the components of \( y_t \) are highly correlated with each other. Fig.19 depicts the cross correlogram of the transformed series \( \tilde{z}_t = \tilde{\Gamma}_y y_t \). We can see clearly from Fig.19 that the six components of \( \tilde{z}_t \) can be divided into 3 groups: \( \{1, 4, 6\} \), \( \{2, 5\} \) and \( \{3\} \), as there are no significant correlations across those three groups.
Table 6: The proportions of correct and incomplete segmentations in Example 5.

<table>
<thead>
<tr>
<th>n</th>
<th>100</th>
<th>200</th>
<th>300</th>
<th>400</th>
<th>500</th>
<th>1000</th>
<th>1500</th>
<th>2000</th>
<th>2500</th>
<th>3000</th>
</tr>
</thead>
<tbody>
<tr>
<td>Correct segmentation</td>
<td>0.350</td>
<td>0.564</td>
<td>0.660</td>
<td>0.712</td>
<td>0.800</td>
<td>0.896</td>
<td>0.906</td>
<td>0.920</td>
<td>0.932</td>
<td>0.940</td>
</tr>
<tr>
<td>Incomplete segmentation</td>
<td>0.508</td>
<td>0.386</td>
<td>0.326</td>
<td>0.282</td>
<td>0.192</td>
<td>0.104</td>
<td>0.094</td>
<td>0.080</td>
<td>0.068</td>
<td>0.060</td>
</tr>
<tr>
<td>Incomplete with $\hat{q} = 2$</td>
<td>0.278</td>
<td>0.258</td>
<td>0.250</td>
<td>0.196</td>
<td>0.150</td>
<td>0.088</td>
<td>0.094</td>
<td>0.080</td>
<td>0.064</td>
<td>0.054</td>
</tr>
</tbody>
</table>

Table 7: The proportions of correct and incomplete segmentations in Example 6.

<table>
<thead>
<tr>
<th>n</th>
<th>400</th>
<th>500</th>
<th>1000</th>
<th>1500</th>
<th>2000</th>
<th>2500</th>
<th>3000</th>
</tr>
</thead>
<tbody>
<tr>
<td>Correct segmentation</td>
<td>0.058</td>
<td>0.118</td>
<td>0.492</td>
<td>0.726</td>
<td>0.862</td>
<td>0.902</td>
<td>0.940</td>
</tr>
<tr>
<td>Incomplete segmentation</td>
<td>0.516</td>
<td>0.672</td>
<td>0.460</td>
<td>0.258</td>
<td>0.130</td>
<td>0.096</td>
<td>0.060</td>
</tr>
<tr>
<td>Incomplete with $\hat{q} = 4$</td>
<td>0.144</td>
<td>0.202</td>
<td>0.242</td>
<td>0.182</td>
<td>0.110</td>
<td>0.088</td>
<td>0.050</td>
</tr>
</tbody>
</table>

Example 6. Now in (2.1) let $p = 20$, $x_t = (x_{1:t}, \ldots, x_{20:t})'$, and

\[
\begin{align*}
  x_{i,t} &= \eta_{t+i-1}^{(1)} \quad (i = 1, \ldots, 6), \\
  x_{i,t} &= \eta_{t+i-7}^{(4)} \quad (i = 7, \ldots, 11), \\
  x_{i,t} &= \eta_{t+i-12}^{(5)} \quad (i = 12, \ldots, 15), \\
  x_{i,t} &= \eta_{t+i-16}^{(6)} \quad (i = 16, 17, 18) \quad \text{and} \quad x_{i,t} = \eta_{t+i-19}^{(7)} \quad (i = 19, 20),
\end{align*}
\]

where $\eta_t^{(1)}$ is the same as in Example 5, and

\[
\begin{align*}
  \eta_t^{(4)} &= -0.4\eta_{t-1}^{(4)} + 0.5\eta_{t-2}^{(4)} + \varepsilon_t^{(4)} + \varepsilon_{t-1}^{(4)} + 0.8\varepsilon_{t-2}^{(4)} + 1.5\varepsilon_{t-3}^{(4)} + 1.8\varepsilon_{t-4}^{(4)}, \\
  \eta_t^{(5)} &= 0.85\eta_{t-1}^{(5)} - 0.3\eta_{t-2}^{(5)} + \varepsilon_t^{(5)} + \varepsilon_{t-1}^{(5)} + 0.5\varepsilon_{t-2}^{(5)} + 1.2\varepsilon_{t-3}^{(5)}, \\
  \eta_t^{(6)} &= 0.8\eta_{t-1}^{(6)} - 0.5\eta_{t-2}^{(6)} + \varepsilon_t^{(6)} + \varepsilon_{t-1}^{(6)} + 0.8\varepsilon_{t-2}^{(6)} + 1.8\varepsilon_{t-3}^{(6)}, \\
  \eta_t^{(7)} &= -0.7\eta_{t-1}^{(7)} - 0.5\eta_{t-2}^{(7)} + \varepsilon_t^{(7)} - \varepsilon_{t-1}^{(7)} - 0.8\varepsilon_{t-2}^{(7)}.
\end{align*}
\]

Now $q = 5$ with the number of components in those 5 independent subseries being, respectively, 6, 5, 4, 3 and 2. For seven different sample sizes between 400 and 3000, the proportions of correct segmentations and incomplete segmentations in the simulation with 500 replications are listed in Table 7. The boxplots of the estimation errors for the instances with the correct segmentation are presented in Fig.21. Comparing with the results for $p = 6$ in Example 5, the performance is worse although the estimation is still accurate especially for the reasonably large $n$. For example when $n = 1500$, the proportion of the correct segmentations is 72.6% in the 500 replications. Plus the instances of the incomplete segmentation with $\hat{q} = q - 1 = 4$, the proportion increases to 90.8%. Thus we can claim that substantial dimension reduction is achieved.

5 Segmenting multivariate volatility processes

The proposed methodology in Section 2 can be readily extended to segment multivariate volatility processes. To this end, let $y_t$ is a $p \times 1$ volatility process. Let $F_t = \sigma(y_t, y_{t-1}, \ldots)$ and
Var(\(y_t|\mathcal{F}_{t-1}\)) = \(\Sigma_y(t)\). There is no loss of generality to assume that \(E(y_t) = 0\) and Var(\(y_t\)) = I_p.

Suppose that there exists an orthogonal matrix \(A\) for which \(y_t = Ax_t\) and

\[\text{Var}(x_t|\mathcal{F}_{t-1}) = \text{diag}\{\Sigma_1(t), \ldots, \Sigma_q(t)\},\]

where \(\Sigma_1(t), \ldots, \Sigma_q(t)\) are, respectively, \(p_1 \times p_1, \ldots, p_q \times p_q\) non-negative definite matrices. Hence the latent \(p\)-dimensional volatility process \(x_t\) can be segmented into \(q\) lower-dimensional processes, and there exist no conditional cross correlations across those \(q\) processes.

To recover the hidden segmentation in volatility, let

\[W_y = \sum_{B \in \mathcal{B}_{t-1}} \left[ E\{y_t y_t' I(B)\} \right]^2, \quad W_x = \sum_{B \in \mathcal{B}_{t-1}} \left[ E\{x_t x_t' I(B)\} \right]^2,\]

where \(\mathcal{B}_{t-1}\) be a \(\pi\)-class and the \(\sigma\)-algebra generated by \(\mathcal{B}_{t-1}\) is equal to \(\mathcal{F}_{t-1}\). Since it holds for any \(B \in \mathcal{B}_{t-1}\) that

\[E\{x_t x_t' I(B)\} = E\{I(B) E(x_t x_t' | \mathcal{F}_{t-1})\} = E[I(B) \text{diag}\{\Sigma_1(t), \ldots, \Sigma_q(t)\}]\]

is a block diagonal matrix, so is \(W_x\). Now (2.6) still holds for the newly defined \(W_y\) and \(W_x\) above. Thus \(A\) can be estimated exactly in the same manner as in Section 2.1. An estimator for \(W_y\) can be defined as

\[\hat{W}_y = \sum_{B \in \mathcal{B}} \sum_{k=1}^{k_0} \left\{ \frac{1}{n-k} \sum_{t=k+1}^{n} y_t y_t' \mathbb{1}(y_{t-k} \in B) \right\}^2,\]

where \(\mathcal{B}\) may consist of \(\{u \in \mathbb{R}^p : \|u\|_2 \leq \|y_t\|_2\}\) for \(t = 1, \ldots, n\). See Fan et al. (2008). We illustrate this idea with a real data example below.

**Example 7.** Fig.22 are time series plots of the daily returns of the stocks of Bank of America Corporation, Dell Inc., JPMorgan Chase&Co., FedEx Corporation, McDonald’s Corp. and American International Group in 2 January 2002 – 10 July 2008. For this data, \(n = 1642\) and \(p = 6\). Denote \(y_t = (y_{1,t}, \ldots, y_{6,t})'\) these 6 returns on the \(t\)-th day. By fitting each return series a GARCH(1,1) model, we calculate the residuals \(\varepsilon_{i,t} = y_{i,t}/\hat{\sigma}_{i,t}\) for \(i = 1, \ldots, 6\), where \(\hat{\sigma}_{i,t}\) denotes the predicted volatility for the \(i\)-th return at time \(t\) based on the fitted GARCH(1,1) model. The cross correlogram of those 6 residual series are plotted in Fig.23, which shows the strong and significant concurrent correlations across all 6 residual series. This indicates clearly that \(\text{Var}(y_t|\mathcal{F}_{t-1})\) is not a block diagonal matrix.

Now we apply the segmentation transform stated above, the transformed series \(\tilde{x}_t = \hat{B} y_t\) are plotted in Fig.24. The transformation matrix \(\hat{B}\) is

\[
\hat{B} = \begin{pmatrix}
-0.227 & -0.093 & 0.031 & 0.550 & 0.348 & -0.041 \\
-0.203 & -0.562 & 0.201 & 0.073 & -0.059 & 0.158 \\
0.022 & 0.054 & -0.068 & 0.436 & -0.549 & 0.005 \\
-0.583 & 0.096 & -0.129 & -0.068 & -0.012 & 0.668 \\
0.804 & -0.099 & -0.409 & -0.033 & 0.008 & 0.233 \\
0.144 & -0.012 & -0.582 & 0.131 & 0.098 & -0.028
\end{pmatrix}.
\]
Now we repeat the whitening process above for $\hat{x}_t$, i.e. fit a GARCH(1,1) model for each of the component series of $\hat{x}_t$ and calculate the residuals. Fig.25 presents the cross correlogram of these 6 new residual series. There exist no significant cross correlations among the 6 residual series. This is the significant evidence to support the assertion that $\text{Var}(x_t|\mathcal{F}_{t-1})$ is a diagonal matrix. For this example, the segmentation method leads to the conditional uncorrelated components of Fan et al. (2008).

6 A final remark

This paper proposes a contemporaneous linear transformation to segment a multiple time series into several both contemporaneously and serially uncorrelated subseries. The method can be used as a preliminary step to reduce a high dimensional time series modelling problem into several lower dimensional problems. The reduction of dimensionality is often substantial and effective.

The method can be viewed as a generalised version of principal component analysis. The difference is that the intended segmentation is not guaranteed to exist. However even in the cases when the segmentation does not always exist, the proposed method provides an approximate segmentation which ignores some minor (though still significant) cross correlations and, therefore, leads to a parsimonious modelling strategy. Those parsimonious strategies are often more effective than, say, direct VAR modelings. See, e.g., Example 3.

7 Appendix

7.1 Proof of Proposition 1

(i) Write $W_x = \text{diag}(W_{x,1}, \ldots, W_{x,q})$. Then all the $q$ blocks $W_{x,1}, \ldots, W_{x,q}$ are positive definite matrices. Performing spectral decomposition for each block, we obtain

$$W_{x,k}\Gamma_k = \Gamma_k \Lambda_k, \quad k = 1, \ldots, q,$$

where $\Gamma_k \Gamma'_k$ is an identity matrix, and $\Lambda_k$ is a diagonal matrix. Now let

$$\Gamma_x = \text{diag}(\Gamma_1, \ldots, \Gamma_q) \quad \text{and} \quad \Lambda = \text{diag}(\Lambda_1, \ldots, \Lambda_q). \quad (7.1)$$

Then (2.7) holds. This completes the proof of (i).

(ii) For any given $D$ in (2.7), we can rearrange the eigenvalues in the main diagonal of $D$ to obtain $\Lambda$ specified in (7.1). Correspondingly, we re-order the columns of $\Gamma_x$ according to the new order of eigenvalues and denote the new matrix by $\tilde{\Gamma}_x$, i.e. the columns of $\tilde{\Gamma}_x$ is a permutation of the columns of $\Gamma_x$. For any $1 \leq i \neq j \leq q$, as $W_{x,i}$ and $W_{x,j}$ do not share the same eigenvalues, the linear space spanned by the eigenvectors of $W_{x,i}$ is uniquely determined by $\Gamma_i$ in (i). Hence, $\tilde{\Gamma}_x$ must be in the form of $\text{diag}(\Gamma_1 H_1, \ldots, \Gamma_q H_q)$ for some $H_j H'_j = I_{p_j}$. Therefore, $\tilde{\Gamma}_x$ satisfies the condition in (i). We complete the proof of (ii).
7.2 A useful lemma

Our main theoretical results are based on the following lemma, which is Theorem 8.1.10 in Golub and Van Loan (1996).

**Lemma 1.** Suppose $B$ and $B + E$ are $s \times s$ symmetric matrices and that

$$Q = (Q_1, Q_2)$$

where $Q_1$ is an $s \times r$ matrix and $Q_2$ is an $s \times (s - r)$ matrix

is an orthogonal matrix such that $\mathcal{M}(Q_1)$ is an invariant subspace for $B$ (that is, $B \cdot \mathcal{M}(Q_1) \subset \mathcal{M}(Q_1)$). Partition the matrices $Q'BQ$ and $Q'EQ$ as follows:

$$Q'BQ = \begin{pmatrix} D_1 & 0 \\ 0 & D_2 \end{pmatrix}$$

$$Q'EQ = \begin{pmatrix} E_{11} & E_{21} \\ E_{21} & E_{22} \end{pmatrix}.$$

If $\text{sep}(D_1, D_2) = \min \lambda \in \sigma(D_1), \mu \in \sigma(D_2) | \lambda - \mu | > 0$, where $\sigma(M)$ denotes the set of eigenvalues of the matrix $M$, and $\|E\|_2 \leq \text{sep}(D_1, D_2)/4$, then there exists a matrix $P \in \mathbb{R}^{(s-r) \times r}$ with

$$\|P\|_2 \leq 4 \frac{\text{sep}(D_1, D_2)}{\|E_{21}\|_2}$$

such that the columns of $\hat{Q}_1 = (Q_1 + Q_2P)(I + P'P)^{-1/2}$ define an orthonormal basis for a subspace that is invariant for $B + E$.

**Remark:** From Lemma 1, we have

$$\|\hat{Q}_1 - Q_1\|_2 = \|\{Q_1 + Q_2P - Q_1(I + P'P)^{1/2}\}(I + P'P)^{-1/2}\|_2$$

$$\leq \|Q_1\{I - (I + P'P)^{1/2}\}\|_2 + \|Q_2P\|_2$$

$$\leq 2\|P\|_2 \leq \frac{8}{\text{sep}(D_1, D_2)}\|E_{21}\|_2 \leq \frac{8}{\text{sep}(D_1, D_2)}\|E\|_2. \tag{7.2}$$

On the other hand,

$$\{D(\mathcal{M}(Q_1), \mathcal{M}(\hat{Q}_1))\}^2 = \frac{1}{r} [\text{tr}\{Q'_1(I_p - \hat{Q}_1\hat{Q}_1')Q_1]\]$$

$$\leq \|Q'_1(I_p - \hat{Q}_1\hat{Q}_1')Q_1\|_2 = \|Q'_1Q_1 - \hat{Q}_1\hat{Q}_1\|_2.$$

With the fact

$$Q'_1(Q_1Q'_1 - \hat{Q}_1\hat{Q}_1')Q_1 = -Q'_1(Q_1 - \hat{Q}_1)(Q_1 - \hat{Q}_1)'Q_1 + (Q_1 - \hat{Q}_1)'(Q_1 - \hat{Q}_1),$$

it yields

$$\{D(\mathcal{M}(Q_1), \mathcal{M}(\hat{Q}_1))\}^2 \leq 2\|Q_1 - \hat{Q}_1\|_2^2.$$

Thus, together with (7.2),

$$D(\mathcal{M}(Q_1), \mathcal{M}(\hat{Q}_1)) \leq \frac{8\sqrt{2}}{\text{sep}(D_1, D_2)}\|E\|_2. \tag{7.3}$$
Meanwhile, assume $\widetilde{Q}_1$ is the orthogonal eigenvectors of $B + E$ that their associated eigenvalues are consistent to the corresponding eigenvalues of $B$ respectively. If there are some identical eigenvalues in $D_1$, then there exists an $r \times r$ orthogonal matrix $U$ such that $\widetilde{Q}_1 = \tilde{Q}_1 U$. Therefore,

$$D(M(Q_1), M(\tilde{Q}_1)) \leq \frac{8\sqrt{2}}{\text{sep}(D_1, D_2)} \|E\|_2.$$ 

This result shows us that the consistent estimation of the linear space may still exist even if the eigenvectors spanned such linear space are not uniquely identified.

### 7.3 Proofs for Section 3.1

**Lemma 2.** Under Conditions 1 and 2, if $p$ is fixed, then $\|\hat{\Sigma}_y(k) - \Sigma_y(k)\|_2 = O_p(n^{-1/2})$ for each $k \leq k_0$.

**Proof:** For each $i, j \in \{1, \ldots, p\}$, without lose of generality, we assume $\mu_i = \mu_j = 0$. Then,

$$\hat{\sigma}_{i,j}^{(k)} - \sigma_{i,j}^{(k)} = \frac{1}{n} \sum_{t=1}^{n-k} \left\{ y_{i,t+k} y_{j,t} - E(y_{i,t+k} y_{j,t}) \right\} - \frac{\bar{y}_{j, \cdot}}{n} \sum_{t=1}^{n-k} y_{i,t+k} - \frac{\bar{y}_{i, \cdot}}{n} \sum_{t=1}^{n-k} y_{j,t}$$

$$+ \frac{n-k}{n} \bar{y}_{i, \cdot} \bar{y}_{j, \cdot} - \frac{k}{n} E(y_{i,t+k} y_{j,t})$$

(7.4)

$$= I_1 + I_2 + I_3 + I_4 + I_5,$$

where $\bar{y}_{i, \cdot} = n^{-1} \sum_{t=1}^{n} y_{i,t}$ and $\bar{y}_{j, \cdot} = n^{-1} \sum_{t=1}^{n} y_{j,t}$. By Davydov inequality, following the same arguments of Lemma 6.1 in Chang et al. (2013a), we have

$$I_1 = O_p(n^{-1/2}), \ I_2 = O_p(n^{-1}), \ I_3 = O_p(n^{-1}), \ I_4 = O_p(n^{-1}), \ I_5 = O_p(n^{-1}).$$

This implies the $(i, j)$-th element of $\hat{\Sigma}_y(k) - \Sigma_y(k)$ is $O_p(n^{-1/2})$. Note that $p$ is fixed, then we complete the proof. \hfill $\Box$

**Lemma 3.** Let Conditions 1 and 2 hold, and $p$ be fixed. Then as $n \to \infty$,

$$\|\hat{W}_y - W_y\|_2 = O_p(n^{-1/2}).$$

**Proof:** When $p$ is fixed, this lemma is a straightforward extension of above lemma. \hfill $\Box$

**Proof of Theorem 1:** Note that $p$ is fixed and any two blocks $W_{x,i}$ and $W_{x,j}$, $1 \leq i < j \leq q$, do not share the same eigenvalues. Then, following the remark below Lemma 1,

$$\sup_{1 \leq j \leq q} D(M(\hat{A}_j), M(A_j)) = O_p(\|\hat{W}_y - W_y\|_2^2) = O_p(n^{-1/2})$$

where the last equality follows from Lemma 3. Hence, we complete the proof. \hfill $\Box$
7.4 Proofs for Section 3.2

Aim to construct the theoretical results proposed in Section 3.2, we need the following lemma given in Corollary 6.1 of Rio (2000).

Lemma 4. Let \( l > 2 \) and \((Z_i)_{i \geq 0}\) be a sequence of real-valued and centered random variables and \((\alpha_k)_{k \geq 0}\) be the sequence of strong mixing coefficients defined via \((Z_i)_{i > 0}\). Denote \( S_k = \sum_{i=1}^{k} Z_i \). Suppose that, for some \( \tau > 0 \), \( \Lambda_{l+\tau}(Z_i) := \sup_{u > 0} |u^{l+\tau} P(|Z_i| > u)|^{1/(l+\tau)} < \infty \) for any positive integer \( i \) and that the strong mixing coefficients satisfy

\[
\alpha_k \leq \phi(k+1)^{-l(l+\tau)/(2\tau)} \quad \text{for any} \ k \geq 0 \quad \text{and some positive} \ \phi.
\]

Then, there exists some positive constant \( C(\phi, l) \) such that

\[
E\left( \sup_{1 \leq k \leq n} |S_k|^l \right) \leq C(\phi, l) \phi^{-l/(l+\tau)} n^{l/2} \left( \sup_{i > 0} \Lambda_{l+\tau}(Z_i) \right)^l.
\]

Based on Lemma 4, we have the following lemma.

Lemma 5. Under Conditions 4 and 5, there exists a positive constant \( C \) only depending on \( l \) and \( \tau \) given in Condition 4 such that

\[
P\left( \max_{i,j} |\bar{\sigma}_{i,j}^{(k)} - \sigma_{i,j}^{(k)}| \geq x \right) \leq C p^2 (x^{-l} + x^{-l/2}) n^{-l/2}
\]

for any \( x > 0 \) such that \( nx \to \infty \) and \( k = 1, \ldots, k_0 \).

Proof: Note that

\[
P\left( \max_{i,j} |\bar{\sigma}_{i,j}^{(k)} - \sigma_{i,j}^{(k)}| \geq x \right) \leq \sum_{i,j=1}^{p} P \left( |\bar{\sigma}_{i,j}^{(k)} - \sigma_{i,j}^{(k)}| \geq x \right).
\]

It is sufficient to bound \( P(|\bar{\sigma}_{i,j}^{(k)} - \sigma_{i,j}^{(k)}| \geq x) \) for each \( i, j = 1, \ldots, p \). Let \( \bar{y}_i = n^{-1} \sum_{t=1}^{n} y_{i,t} \) and \( \bar{y}_j = n^{-1} \sum_{t=1}^{n} y_{j,t} \). Without lose of generality, we assume \( \mu_i = \mu_j = 0 \). From Condition 4, it yields

\[
\sup_{t} \sup_{i,j} P \left\{ |y_{i,t+k}y_{j,t} - E(y_{i,t+k}y_{j,t})| > x \right\} = O(x^{-l}) \quad \text{as} \ x \to \infty.
\]

By Lemma 4 and Markov inequality, it yields that

\[
P\left( \left| \frac{1}{n} \sum_{t=1}^{n-k} \{y_{i,t+k}y_{j,t} - E(y_{i,t+k}y_{j,t})\} \right| \geq x \right) \leq C x^{-l} n^{-l/2} \quad \text{for any} \ x > 0.
\]

(7.5)

Meanwhile, note that

\[
P\left( \left| \frac{n-k}{n} \sum_{t=1}^{n-k} y_{i,t} \right| \geq x \right) \leq P\left( |\bar{y}_i| \geq x^{1/2} \right) + P\left( \left| \frac{1}{n} \sum_{t=1}^{n-k} y_{j,t} \right| \geq x^{1/2} \right),
\]

using the same argument above, we can obtain

\[
P\left( \left| \frac{n-k}{n} \sum_{t=1}^{n-k} y_{j,t} \right| \geq x \right) \leq C x^{-l/2} n^{-l/2} \quad \text{for any} \ x > 0.
\]

(7.6)

Noting (7.4), together with (7.5) and (7.6), we complete the proof of this lemma. \( \Box \)
Lemma 6. Under Condition 3, it holds that
\[
\max_j \sum_{i=1}^{p} |\sigma_{i,j}^{(k)}|^{t} \leq (2S_{\max} + 1)s_1 s_2 \quad \text{and} \quad \max_i \sum_{j=1}^{p} |\sigma_{i,j}^{(k)}|^{t} \leq (2S_{\max} + 1)s_1 s_2;
\]
for \(i\) given in Condition 3 and \(S_{\max} = \max_{1 \leq j \leq q} p_j\).

Proof: Write \(\Sigma_x(k) = (\sigma_{x,i,j}^{(k)})_{i,j=1,\ldots,p} \) and \(A = (a_1, \ldots, a_p)^t\). As
\[
\sigma_{i,j}^{(k)} = a_1 \Sigma_x(k) a_j = \sum_{v,l=1}^{p} a_{i,v} \sigma_{x,v,l}^{(k)} a_{j,l} = \sum_{|v-l| \leq S_{\max}} a_{i,v} \sigma_{x,v,l}^{(k)} a_{j,l},
\]
then by the inequality \((x + y)^t \leq x^t + y^t\) for any \(x \geq 0\) and \(y \geq 0\), it yields
\[
\sum_{i=1}^{p} |\sigma_{i,j}^{(k)}|^{t} \leq \sum_{|v-l| \leq S_{\max}} \sum_{i=1}^{p} |a_{i,v}|^t |a_{j,l}|^t \|\Sigma_x(k)\|_{\infty}^t \leq s_1 \|\Sigma_x(k)\|_{\infty}^t \sum_{|v-l| \leq S_{\max}} |a_{j,l}|^t \leq (2S_{\max} + 1)s_1 \|\Sigma_x(k)\|_{\infty} \sum_{l=1}^{p} |a_{j,l}|^t \leq (2S_{\max} + 1)s_1 s_2.
\]
Note that above inequality holds for any \(j\), hence we complete the proof of the first result in this lemma. Using the same argument, we can prove the second one. \(\square\)

Lemma 7. Under Conditions 3, 4 and 5, for any \(k \leq k_0\), it holds that
\[
\|T_u(\Sigma_y(k)) - \Sigma_y(k)\|_2 = O_p\{ (p^{4/5}n^{-1})^{(1-\epsilon)/3} \delta \}
\]
provided that \(p = o(n^{1/4})\), where \(\delta\) is defined in (3.7).

Proof: Note that
\[
\|T_u(\Sigma_y(k)) - \Sigma_y(k)\|_2 \leq \|T_u(\Sigma_y(k)) - \Sigma_y(k)\|_2 + \|T_u(\Sigma_y(k)) - T_u(\Sigma_y(k))\|_2.
\]
For the first part, we have
\[
\|T_u(\Sigma_y(k)) - \Sigma_y(k)\|_2 \leq \left[ \max_i \sum_{j=1}^{p} |\sigma_{i,j}^{(k)}| \mathbb{I}\{|\sigma_{i,j}^{(k)}| < u\} \right]^{1/2} \left[ \max_j \sum_{i=1}^{p} |\sigma_{i,j}^{(k)}| \mathbb{I}\{|\sigma_{i,j}^{(k)}| < u\} \right]^{1/2}.
\]
\(\epsilon\)From Lemma 6,
\[
\sum_{j=1}^{p} |\sigma_{i,j}^{(k)}| \mathbb{I}\{|\sigma_{i,j}^{(k)}| < u\} \leq u^{-\epsilon} \sum_{j=1}^{p} |\sigma_{i,j}^{(k)}|^{t} \leq u^{-\epsilon}(2S_{\max} + 1)s_1 s_2.
\]
Similarly, \(\sum_{i=1}^{p} |\sigma_{i,j}^{(k)}| \mathbb{I}\{|\sigma_{i,j}^{(k)}| < u\} \leq u^{-\epsilon}(2S_{\max} + 1)s_1 s_2\). Hence,
\[
\|T_u(\Sigma_y(k)) - \Sigma_y(k)\|_2 \leq u^{-\epsilon}(2S_{\max} + 1)s_1 s_2.
\]
(7.7)
On the other hand, we have
\[
\|T_u(\hat{\Sigma}_y(k)) - T_u(\Sigma_y(k))\|_2 \leq \left[ \max_j \sum_{i=1}^p |\hat{\sigma}_{i,j}^{(k)}| I\{ |\hat{\sigma}_{i,j}^{(k)}| \geq u, |\sigma_{i,j}^{(k)}| \geq u \} \right]^{1/2} \\
\times \left[ \max_j \sum_{i=1}^p |\hat{\sigma}_{i,j}^{(k)}| I\{ |\hat{\sigma}_{i,j}^{(k)}| \geq u \} - |\sigma_{i,j}^{(k)}| I\{ |\sigma_{i,j}^{(k)}| \geq u \} \right]^{1/2} \\
= : I_6^{1/2} \times I_7^{1/2}.
\] (7.8)

By Triangle inequality,
\[
I_6 \leq \max_j \sum_{i=1}^p |\hat{\sigma}_{i,j}^{(k)} - \sigma_{i,j}^{(k)}| I\{ |\hat{\sigma}_{i,j}^{(k)}| \geq u, |\sigma_{i,j}^{(k)}| \geq u \} \\
+ \max_j \sum_{i=1}^p |\hat{\sigma}_{i,j}^{(k)}| I\{ |\hat{\sigma}_{i,j}^{(k)}| > u, |\sigma_{i,j}^{(k)}| < u \} + \max_j \sum_{i=1}^p |\hat{\sigma}_{i,j}^{(k)}| I\{ |\hat{\sigma}_{i,j}^{(k)}| < u, |\sigma_{i,j}^{(k)}| \geq u \} \\
= : I_{6,1} + I_{6,2} + I_{6,3}.
\]

Let \( Z = \max_{i,j} |\hat{\sigma}_{i,j}^{(k)} - \sigma_{i,j}^{(k)}| \). For \( I_{6,1} \), by Lemma 6,
\[
I_{6,1} \leq \max_{i,j} |\hat{\sigma}_{i,j}^{(k)} - \sigma_{i,j}^{(k)}| \times \max_j \sum_{i=1}^p I\{ |\hat{\sigma}_{i,j}^{(k)}| \geq u, |\sigma_{i,j}^{(k)}| \geq u \} \\
\leq \max_{i,j} |\hat{\sigma}_{i,j}^{(k)} - \sigma_{i,j}^{(k)}| \times \max_j \sum_{i=1}^p |\sigma_{i,j}^{(k)}|^t u^{-t} \leq Z u^{-t} (2S_{\text{max}} + 1)s_1 s_2.
\]

For \( I_{6,2} \), by Triangle inequality and Lemma 6,
\[
I_{6,2} \leq \max_j \sum_{i=1}^p |\hat{\sigma}_{i,j}^{(k)} - \sigma_{i,j}^{(k)}| I\{ |\hat{\sigma}_{i,j}^{(k)}| > u, |\sigma_{i,j}^{(k)}| < u \} + \max_j \sum_{i=1}^p |\sigma_{i,j}^{(k)}| I\{ |\sigma_{i,j}^{(k)}| < u \} \\
\leq \max_j \sum_{i=1}^p |\hat{\sigma}_{i,j}^{(k)} - \sigma_{i,j}^{(k)}| I\{ |\hat{\sigma}_{i,j}^{(k)}| \geq u, |\sigma_{i,j}^{(k)}| < u \} + u^{-t} (2S_{\text{max}} + 1)s_1 s_2.
\]

We consider the first term in the right-hand side of above inequality. Taking \( \theta \in (0, 1) \), by Triangle inequality and Lemma 6, then
\[
\max_j \sum_{i=1}^p |\hat{\sigma}_{i,j}^{(k)} - \sigma_{i,j}^{(k)}| I\{ |\hat{\sigma}_{i,j}^{(k)}| \geq u, |\sigma_{i,j}^{(k)}| < u \} \\
\leq \max_j \sum_{i=1}^p |\hat{\sigma}_{i,j}^{(k)} - \sigma_{i,j}^{(k)}| I\{ |\hat{\sigma}_{i,j}^{(k)}| \geq u, \theta u \} \\
+ \max_j \sum_{i=1}^p |\hat{\sigma}_{i,j}^{(k)} - \sigma_{i,j}^{(k)}| I\{ |\hat{\sigma}_{i,j}^{(k)}| \geq u, \theta u < |\sigma_{i,j}^{(k)}| < u \} \\
\leq Z \times \max_j \sum_{i=1}^p I\{ |\hat{\sigma}_{i,j}^{(k)} - \sigma_{i,j}^{(k)}| \geq (1 - \theta) u \} + Z \times \max_j \sum_{i=1}^p |\sigma_{i,j}^{(k)}|^t (\theta u)^{-t} \\
\leq Z \times \max_j \sum_{i=1}^p I\{ |\hat{\sigma}_{i,j}^{(k)} - \sigma_{i,j}^{(k)}| \geq (1 - \theta) u \} + Z \theta^{-t} u^{-t} (2S_{\text{max}} + 1)s_1 s_2.
Thus,

\[ I_{6,2} \leq Z \times \max_j \sum_{i=1}^p \mathbb{I}\{|\tilde{\sigma}_{i,j}^{(k)} - \sigma_{i,j}^{(k)}| \geq (1-\theta)u\} + Z\theta^{-i}u^{-i}(2S_{\text{max}} + 1)s_1s_2 + u^{1-i}(2S_{\text{max}} + 1)s_1s_2. \]

On the other hand, by Triangle inequality and Lemma 6,

\[ I_{6,3} \leq \max_{i,j} \sum_{k=1}^p |\tilde{\sigma}_{i,j}^{(k)} - \sigma_{i,j}^{(k)}| \mathbb{I}\{|\tilde{\sigma}_{i,j}^{(k)}| < u, |\sigma_{i,j}^{(k)}| \geq u\} + \max_{i,j} \sum_{k=1}^p |\tilde{\sigma}_{i,j}^{(k)}| \mathbb{I}\{|\tilde{\sigma}_{i,j}^{(k)}| < u, |\sigma_{i,j}^{(k)}| \geq u\} \]

\[ \leq \max_{i,j} |\tilde{\sigma}_{i,j}^{(k)} - \sigma_{i,j}^{(k)}| \times \max_{i,j} \sum_{k=1}^p \mathbb{I}\{|\sigma_{i,j}^{(k)}| \geq u\} + u \max_{i,j} \sum_{k=1}^p \mathbb{I}\{|\sigma_{i,j}^{(k)}| \geq u\} \]

\[ \leq \max_{i,j} |\tilde{\sigma}_{i,j}^{(k)} - \sigma_{i,j}^{(k)}| \times u^{-i} \max_{j} \sum_{k=1}^p |\sigma_{i,j}^{(k)}|^i + u^{1-i} \max_{j} \sum_{k=1}^p |\sigma_{i,j}^{(k)}|^i \]

\[ \leq Zu^{-i}(2S_{\text{max}} + 1)s_1s_2 + u^{1-i}(2S_{\text{max}} + 1)s_1s_2. \]

Hence,

\[ I_6 \leq 2Zu^{-i}(2S_{\text{max}} + 1)s_1s_2 + Z\theta^{-i}u^{-i}(2S_{\text{max}} + 1)s_1s_2 + 2u^{1-i}(2S_{\text{max}} + 1)s_1s_2 \]

\[ + Z \times \max_j \sum_{i=1}^p \mathbb{I}\{|\tilde{\sigma}_{i,j}^{(k)} - \sigma_{i,j}^{(k)}| \geq (1-\theta)u\}. \] (7.9)

Similarly, we have

\[ I_7 \leq 2Zu^{-i}(2S_{\text{max}} + 1)s_1s_2 + Z\theta^{-i}u^{-i}(2S_{\text{max}} + 1)s_1s_2 + 2u^{1-i}(2S_{\text{max}} + 1)s_1s_2 \]

\[ + Z \times \max_i \sum_{j=1}^p \mathbb{I}\{|\tilde{\sigma}_{i,j}^{(k)} - \sigma_{i,j}^{(k)}| \geq (1-\theta)u\}. \] (7.10)

For any \( \lambda > 1 \), noting \( u = Mp^{2l/n-1/2} \), by Markov inequality and Lemma 5,

\[ P\left[ \max_j \sum_{i=1}^p \mathbb{I}\{|\tilde{\sigma}_{i,j}^{(k)} - \sigma_{i,j}^{(k)}| \geq (1-\theta)Mp^{2l/n-1/2}\} \geq \lambda \right] \]

\[ \leq \lambda^{-1} \sum_{i,j=1}^p P\left(|\tilde{\sigma}_{i,j}^{(k)} - \sigma_{i,j}^{(k)}| \geq (1-\theta)Mp^{2l/n-1/2}\right) \]

\[ \leq C\lambda^{-1}(1-\theta)^{-l}M^{-l} + C\lambda^{-1}(1-\theta)^{-l/2}M^{-l/2}pn^{-l/4} \]

which implies

\[ \max_j \sum_{i=1}^p \mathbb{I}\{|\tilde{\sigma}_{i,j}^{(k)} - \sigma_{i,j}^{(k)}| \geq (1-\theta)Mp^{2l/n-1/2}\} = O_p(1) \]

provided that \( p = o(n^{1/4}) \). From Lemma 5, it yields \( Z = O_p(p^{2l/n-1/2}) \). By (7.9) and (7.10), we have

\[ I_6 = O_p\{ (p^{4l/n-1})^{(1-\delta)/2} \} = I_7. \]

Therefore, from (7.7) and (7.8), we have

\[ \| T_u(\tilde{\Sigma}_{y}(k)) - \Sigma_{y}(k) \|_2 = O_p\{ (p^{4l/n-1})^{(1-\delta)/2} \}. \]

Hence, we complete the proof of this lemma. \[ \square \]
Lemma 8. Let Conditions 3, 4 and 5 hold. As \( n \to \infty \) and \( p = o(n^{1/4}) \) with \( l \) given in Condition 4, it holds that
\[
\| \mathbf{\hat{W}}_y^{(\text{thre})} - \mathbf{W}_y \|_2 = O_p\{ \kappa (p^{4/l} n^{-1})^{(1-i)/2} \delta + (p^{4/l} n^{-1})^{1-i} \delta^2 \},
\]
where \( \delta \) and \( \kappa \) are defined in (3.7).

Proof: Note that
\[
\| T_u(\mathbf{\hat{\Sigma}}_y(k))T_u(\mathbf{\Sigma}_y(k)') - \mathbf{\Sigma}_y(k)\mathbf{\Sigma}_y(k)' \|_2 \\
\leq 2 \| T_u(\mathbf{\hat{\Sigma}}_y(k)) - \mathbf{\Sigma}_y(k) \|_2 \| \mathbf{\Sigma}_y(k) \|_2 + \| T_u(\mathbf{\hat{\Sigma}}_y(k)) - \mathbf{\Sigma}_y(k) \|_2 \\
\leq 2 \| T_u(\mathbf{\hat{\Sigma}}_y(k)) - \mathbf{\Sigma}_y(k) \|_2 \| \mathbf{\Sigma}_x(k) \|_2 + \| T_u(\mathbf{\hat{\Sigma}}_y(k)) - \mathbf{\Sigma}_y(k) \|_2.
\]

From Lemma 7, we have
\[
\| T_u(\mathbf{\hat{\Sigma}}_y(k))T_u(\mathbf{\Sigma}_y(k)') - \mathbf{\Sigma}_y(k)\mathbf{\Sigma}_y(k)' \|_2 = O_p\{ \kappa (p^{4/l} n^{-1})^{(1-i)/2} \delta + (p^{4/l} n^{-1})^{1-i} \delta^2 \}.
\]
As \( k_0 \) is fixed, we complete the proof of Lemma 8.

Proof of Theorem 2: Same as the proof of Theorem 1, we can prove the result.

7.5 Proofs for Section 3.3

Lemma 9. Under Conditions 6 and 7, let \( \gamma_1^{-1} = 2r_1^{-1} + r_2^{-1} \) and \( \gamma_2^{-1} = r_1^{-1} + r_2^{-1} \), there exists a positive constant \( C \) such that for any \( k \leq k_0 \),
\[
P\left( \max_{i,j} \left| \hat{\theta}_{i,j}^{(k)} - \sigma_{i,j}^{(k)} \right| \geq s \right) \leq C p^2 n \exp(-Cs^{\gamma_1} n^{\gamma_1}) + C p^2 n \exp(-Cs^{\gamma_2} n^{\gamma_2}) \\
+ C p^2 \exp(-Cs^2 n) + C p^2 \exp(-Cs n)
\]
for any \( s > 0 \) such that \( ns \to \infty \).

Proof: For any \( i,j \in \{1,\ldots,p\} \), without lose of generality, we assume \( \mu_i = \mu_j = 0 \). Under Condition 6, by the same argument of Lemma 2 in Chang et al. (2013b),
\[
P\{ |y_{i,t+k}y_{j,t} - E(y_{i,t+k}y_{j,t})| > s \} \leq 2K_1 \exp(-K_2 s^{\gamma_2}/2) \quad \text{for any} \ s > 0.
\]
In rest of the proof, we will use \( C \) to denote a general positive constant which does not depend on \( i,j \) and \( k \) and that may be different in different uses. By the theorem of Merlevède et al. (2011), for any \( s > 0 \) such that \( ns \to \infty \),
\[
P( |I_1| \geq s ) \leq C n \exp(-Cs^{\gamma_1} n^{\gamma_1}) + C \exp(-Cs^2 n),
\]
\[
P( |I_2| \geq s ) \leq C n \exp(-Cs^{\gamma_2} n^{\gamma_2}) + C \exp(-Cs n),
\]
\[
P( |I_3| \geq s ) \leq C n \exp(-Cs^{\gamma_2} n^{\gamma_2}) + C \exp(-Cs n),
\]
\[
P( |I_4| \geq s ) \leq C n \exp(-Cs^{\gamma_2} n^{\gamma_2}) + C \exp(-Cs n),
\]
where $I_1, I_2, I_3$ and $I_4$ are given in (7.4). Hence, by (7.4), for any $s > 0$ such that $ns \rightarrow \infty$,

\[ P(|\hat{\sigma}_{i,j}^{(k)} - \sigma_{i,j}^{(k)}| > s) \leq Cn \exp(-Cs^{\gamma_1}n^{\gamma_1}) + Cn \exp(-Cs^{\gamma_2/2}n^{\gamma_2}) + C \exp(-Cs^2n) + C \exp(-Cs n). \]

By Bonferroni inequality, we obtain the result.

**Lemma 10.** Let Conditions 3, 6 and 7 hold, and $p$ be specified in (3.8). It holds that

\[ \|\hat{W}_y^{(\text{the})} - W_y\|_2 = O_p\{\kappa(n^{-1} \log p)(1-\iota)/2 \delta + (n^{-1} \log p)^{1-\iota}\delta^2\}, \]

where $\delta$ and $\kappa$ are defined in (3.7).

**Proof:** By the same arguments of Lemma 7, it yields that

\[ \|T_u(\hat{\Sigma}_y(k)) - \Sigma_y(k)\|_2 = O_p\{(n^{-1} \log p)^{(1-\iota)/2} \delta\} \]

provided that $\log p = o(n^{\gamma_1/(2-\gamma_1)})$. Following the same arguments of Lemma 8, we have

\[ \|T_u(\hat{\Sigma}_y(k))T_u(\Sigma_y(k)') - \Sigma_y(k)\Sigma_y(k)\|_2 = O_p\{\kappa(n^{-1} \log p)^{(1-\iota)/2} \delta + (n^{-1} \log p)^{1-\iota}\delta^2\}. \]

Hence, we obtain the results.

**Proof of Theorem 3:** Same as the proof of Theorem 1, we have the results.

**References**


Figure 1: Time series plots of monthly temperatures in January 1954 - December 1998 in 7 cities (from top to bottom, Nanjing, Dongtai, Huoshan, Hefei, Shanghai, Anqing and Hangzhou) in Eastern China.

Figure 2: Cross correlogram of the 7 temperature series plotted in Fig.1.
Figure 3: Time series resulted from the segmentation transformation from the 7 monthly temperatures series plotted in Fig.1.

Figure 4: Cross correlogram of the 7 transformed series plotted in Fig.3, indicating the segmentation: \{1, 2, 3\}, \{4\}, \{5\}, \{6\} and \{7\}.
Figure 5: Time series plots of 8 differenced monthly US Industrial Indices in January 1947 - December 1993.

Figure 6: Cross correlogram of the 8 differenced industrial index series plotted in Fig.5.
Figure 7: Time series resulted from the segmentation transformation from the 8 differenced monthly US Industrial Indices plotted in Fig.5.

Figure 8: Cross correlogram of the 8 transformed series plotted in Fig.7. The 8 transformed series are segmented into 5 groups: \{1, 2, 3\}, \{4, 8\}, \{5\}, \{6\} and \{7\}. 

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Figure 9: Time series plots of weekly recorded cases of measles in a pre-vaccination period 1948-1965) in 7 cities in England (from top to bottom: London, Bristol, Liverpool, Manchester, Newcastle, Birmingham and Sheffield).

Figure 10: Cross correlogram of the 7 measles series plotted in Fig.9.
Figure 11: Time series resulted from the segmentation transformation from the 7 measles series plotted in Fig.9.

Figure 12: Cross correlogram of the 7 transformed series plotted in Fig.11.
Figure 13: (a) The maximum cross correlations, plotted in descending order, among each of the \( C_i \) = 21 pairs component series of the transformed and prewhitened measles series. The maximization was taken over the lags between -20 to 20. (b) The ratios of two successive correlations in (a).

Figure 14: Time series plots of the daily log-sales of a brand clothes in 8 provinces (from top to button, Beijing, Fujian, Guangdong, Guangxi, Hainan, Hebei, Henan and Hubei) in China.
Figure 15: Cross correlogram of the 8 sales series plotted in Fig.14.

Figure 16: Time series of 8 selected transformed log sales of a brand clothes. The 15th and 16th component series are displayed in the first two panels. The other six series are randomly selected from the remaining 23 transformed series.
Figure 17: Cross correlogram of the 8 transformed series plotted in Fig.16.

Figure 18: Cross correlogram of the $y_t$ in Example 5, $n = 3000$. 
Figure 19: Cross correlogram of the transformed series $\hat{z}_t = \hat{\Gamma}_y y_t$ in Example 5. The components of $\hat{z}_t$ can be segmented into 3 groups: $\{1, 4, 6\}$, $\{2, 5\}$ and $\{3\}$.

Figure 20: The boxplots of estimation errors $\bar{D}(A, \hat{A})$ in Example 5.

Figure 21: The boxplots of the estimated $\bar{D}(\mathcal{M}(A), \mathcal{M}(\hat{A}))$ in Example 6.
Figure 22: Time series plots of the daily returns of the stocks (from top to bottom) Bank of America Corporation, Dell Inc., JPMorgan Chase&Co., FedEx Corporation, McDonald’s Corp. and American International Group in 2 January 2002 – 10 July 2008.

Figure 23: Cross correlogram of the residuals resulted from fitting each series in Fig.22 a GARCH(1,1) model. There are strong concurrent correlations across all the 6 residual series.
Figure 24: Time series plots of the transformed series of 6 daily returns plotted in Fig.22.

Figure 25: Cross correlogram of the residuals resulted from fitting each series in Fig.24 a GARCH(1,1) model. There are no significant correlations across the 6 residual series.