THE SOCIAL COST-OF-LIVING: WELFARE FOUNDATIONS AND ESTIMATION

Thomas F. Crossley
Krishna Pendakur

The Social Cost-of-Living: Welfare Foundations and Estimation[1]

Thomas F. Crossley   Krishna Pendakur

McMaster University   Simon Fraser University

June 20, 2006

Executive Summary

- "How has the cost-of-living changed?" is among the first questions that policy makers and the public ask of economists. One reason is that a vast amount of public expenditure is tied to measured changes in the cost-of-living. For example, many public pensions are indexed to measures of the overall or "social" cost-of-living. While economists have a well developed theory of the cost-of-living for a person, they do not have similarly well developed theory for the cost-of-living for a society.

- If preferences and budgets are identical across people, then the cost-of-living index is identical across people, and there is no problem in identifying the social cost-of-living index. However, if preferences or budgets are heterogeneous across people (as they clearly are), then different people experience different changes in the cost-of-living. Aggregating heterogeneous individual cost-of-living indices into a sensible social cost-of-living index is a difficult problem.

- In this paper we present a new social cost-of-living index that aggregates the cost-of-living indices of heterogeneous individuals. Our new social cost-of-living index answers the following question: What single scaling to everyone's expenditure would hold social welfare constant across a price change? We call this the *common-scaling social cost-of-living index.*

- The common-scaling social cost-of-living index has social welfare foundations, and allows the investigator to easily choose the weight placed on rich and poor households. It is easy to implement, and we have provided methods for estimating both first- and second-order approximations to the index. The latter capture substitution effects. Our estimation methods allow for unobserved preference heterogeneity.

- Finally, the CS-SCOLI has as special cases objects that are either identical or very similar to all the commonly used social cost-of-living indices. This is important. In our framework there is a social welfare function and equivalence scale which lead to the Current Price Index (CPI). Thus, the CPI is given an explicit social welfare foundation. Moreover, an investigator that finds the social welfare function and equivalence scale corresponding to the CPI unpalatable can easily generate a SCOLI more to her tastes.

- To illustrate, we consider changes in the social cost-of-living in the U.S. between 1988 and 2000. We find that both the weighting of rich and poor and the incorporation of second order effects have some impact on our assessment of changes in the social cost-of-living.

Abstract

We present a new class of social cost-of-living indices and a nonparametric framework for estimating these and other social cost-of-living indices. Common social cost-of-living indices can be understood as aggregator functions of approximations of individual cost-of-living indices. The Consumer Price Index (CPI) is the expenditure-weighted average of first-order approximations of each individual's cost-of-living index. This is troubling for three reasons. First, it has not been shown to have a welfare economic foundation for the case where agents are heterogeneous (as they clearly are.) Second, it uses an expenditure-weighted average which downweights the experience of poor households relative to rich households. Finally, it uses only first-order approximations of each individual's cost-of-living index, and thus ignores substitution effects.

We propose a "common-scaling" social cost-of-living index, which is defined as the single scaling to everyone's expenditure which holds social welfare constant across a price change. Our approach has an explicit social welfare foundation and allows us to choose the weights on the costs of rich and poor households. We also give a unique solution for the welfare function for the case where the weights are independent of household expenditure. A first order approximation of our social cost-of-living index nests as special cases commonly used indices such as the CPI. We also provide a nonparametric method for estimating second-order approximations (which account for substitution effects).

JEL Classifications: D11, D12, D63, E31

Keywords: Inflation, Social cost-of-living, Demand, Average Derivatives

1 Introduction

"How has the cost-of-living changed?" is among the first questions that policy makers and the public ask of economists. One reason is that a vast amount of public expenditure is tied to measured changes in the cost-of-living. For example, many public pensions are indexed to measures of the overall or "social" cost-of-living. While economists have a well developed theory of the cost-of-living for a person, they do not have similarly well developed theory for the cost-of-living for a society. If preferences and budgets are identical across people, then the cost-of-living index is identical across people, and there is no problem in identifying the social cost-of-living index. However, if preferences or budgets are heterogeneous across people (as they clearly are), then different people experience different changes in the cost-of-living. In this paper we present a new class of social cost-of-living indices. These indices aggregate the cost-of-living indices of heterogeneous individuals. In addition, we offer a nonparametric framework for estimating these and other social cost-of-living indices.

Most social cost-of-living indices in use—such as the Consumer Price Index (CPI)—can be understood as aggregator functions of approximations of household cost-of-living indices (see, e.g., Prais 1958 or Nicholson 1975 and, especially, Diewert's 1998 overview). The CPI is the expenditure-weighted average of first-order approximations of each individual's cost-of-living index. It is troubling for three reasons.[1] First, it has not been shown to have a welfare economic foundation for the case where agents are heterogeneous. Second, the CPI uses an expenditure-weighted average which down-weights the experience of poor households relative to rich households (and thus is sometimes called a "plutocratic" index). Finally, it uses only first-order approximations of each individual's cost-of-living index, and thus ignores substitution effects.

Many researchers have used an alternative, called the "democratic index", equal to the arithmetic mean of household cost-of-living indices (recent work includes: Kokoski 2000;

---

[1]There are, of course, other problems involved in measuring changes the cost-of-living, including the arrival of new goods, unobserved price heterogeneity and quality change. See Boskin (1998) and Diewert (1998) for surveys.

Crawford and Smith 2002; Ley 2000, 2005). This addresses our second concern, but not the other two. Indeed, the difficulty of aggregating heterogeneous individual cost-of-living indices into a sensible social cost-of-living index has led some observers to suggest abandoning the goal of a social cost-of-living index, and focussing instead on axiomatic approaches to measuring price change (Deaton 1998).

We are more optimistic. Our social cost-of-living index has an explicit social welfare foundation and allows us to choose the weights on rich and poor households. It nests as special cases commonly used indices such as the CPI. We also provide a nonparametric method for estimating second-order approximations (which account for substitution effects).

For an individual, the change in the cost-of-living is the scaling of expenditure required to hold utility constant over a price change. Again, for any given price change, there is heterogeneity across individuals in their cost-of-living changes because preferences and budgets differ across people. Our new social cost-of-living index is the answer to the following question. What single scaling to everyone's expenditure would hold social welfare constant across a price change? We call this the "common-scaling" social cost-of-living index.

With this approach, the inequality-aversion of the social welfare function determines the weights placed on the cost-of-living changes of rich and poor individuals. A first-order approximation is easily derived and nests commonly used indices. In particular, the CPI results from our approach if the (indirect) social welfare function is linear and therefore neutral to inequality. Alternatively, an index similar to the democratic index results if the welfare function is linear in the log of expenditure and thus inequality-averse. Further, we that this index is the unique common-scaling cost-of-living index for the case where the weights are independent of household expenditure (as they are in the democratic index).

Our method for estimating second-order approximations relies on nonparametric estimates of average derivatives. It is similar in spirit to that proposed by Deaton and Ng (1998) for evaluating tax reforms. However, while they estimate a column of uncompensated price effects, we estimate the entire matrix of average compensated prices semi-elasticities

(Slutsky terms). Lewbel (2001) showed that, in the presence of preference heterogenity, the sample average of the matrix of compensated price semi-elasticities is not a consistent estimator of the matrix of average compensated price semi-elasticities. We propose a new estimator of this matrix which exploits the symmetry of the Slutsky matrix and circumvents Lewbel's problem.

To illustrate, we consider changes in the cost-of-living in the U.S. between 1988 and 2000. We find that both the weighting of rich and poor and the incorporation of second order effects have some impact on our assessment of changes in the cost-of-living.

The remainder of the paper is organized as follows. We first outline the theory of the cost-of-living for individuals and propose a social cost-of-living index that aggregates the heterogeneous cost-of-living indices of individuals. Next, we show that commonly used indices, such as the plutocratic and democratic indices, are cases of our general approach. We then show how to nonparametrically estimate second-order approximations of our family of social cost-of-living indices, which includes the plutocratic and democratic cases. Finally, we estimate various social cost-of-living indices with U.S. price and expenditure data.

2   Theory

2.1   Individual Cost-of-Living Index

The standard theory of the cost-of-living for a person is as follows. Let $i = 1, ..., N$ index individuals, each of whom lives in a household with one or more members. For each individual, the number n_i gives the number of members in that person's household. Each individual has an expenditure level x_i equal to total expenditure of that individual's household.

Let $u = V(\mathbf{p}, x, z)$ be the indirect utility function which gives the utility level for an individual living in a household with a T-vector of demographic or other characteristics z, expenditure x and facing prices p. Let $x = C(\mathbf{p}, u, \mathbf{z})$ be the cost function, which is the inverse of V over x. Let $\mathbf{w}$ be the expenditure-share vector, with a subscript for household

or individual.

Many calculations are done at the household level, rather than the individual. For household-level calculations, let $h = 1, ..., H$ index households, let x_h be the total expenditure, n_h be the number of members, and $\mathbf{z}_h$ be the characteristics of household h. Assume that all members of a given household attain the same utility level, and consequently have the same cost-of-living index. We consider environments where expenditure levels and characteristics vary across households, but not within households, and where price vectors are common across all individuals/households.

We define the individual's cost-of-living index (COLI), π_i, as the scaling to expenditure x_i which equates utility at two different price vectors, $\mathbf{p}_0$ and $\mathbf{p}_1$. Formally, we solve

$$V(\mathbf{p}_0, x_i, \mathbf{z}_i) = V(\mathbf{p}_1, \pi_i x_i, \mathbf{z}_i) \tag{1}$$

for π_i. Denoting $x_i = C(\mathbf{p}_0, u_i, \mathbf{z}_i)$ and $u_i = V(p_0, x_i, \mathbf{z}_i)$, the solution may be written in terms of cost functions as

$$\pi_i = C(\mathbf{p}_1, u_i, \mathbf{z}_i)/C(\mathbf{p}_0, u_i, \mathbf{z}_i) = C(\mathbf{p}_1, u_i, \mathbf{z}_i)/x_i.$$

For a household-level calculation, we note that $\pi_i = \pi_h$ for all i in household h. Although most previous work is motivated with household-level calculations, the welfarist framework that we employ below necessitates an individual-level analysis. When all household members are identical, and thus have the same COLI, moving between these levels of analysis is straightforward, and essentially amounts to reweighting.

2.2 Previous Approaches to the Social Cost-of-Living

Since the COLI is different for individuals with different x and $\mathbf{z}$, a social cost-of-living index (SCOLI) must somehow aggregate these individual COLIs. The most commonly used SCOLI is the so-called plutocratic index, Π^P, which is defined as a weighted average of individual

COLIs given by

$$\Pi^P = \frac{1}{\sum_{h=1}^{H} x_h} \sum_{h=1}^{H} x_h \pi_h = \frac{1}{\sum_{i=1}^{N} \frac{x_i}{n_i}} \sum_{i=1}^{N} \frac{x_i}{n_i} \pi_i. \tag{2}$$

The index assigns the household expenditure weight to each household-specific COLI, or, equivalently, assigns the household per-capita expenditure weight to each person-specific COLI. The plutocratic index is used by many statistical agencies, primarily because a first-order approximation to this index is computable without estimation of a demand system and using only aggregate data. In particular, this approximation of Π^P is given by the weighted average of price changes, where the weights are aggregate expenditure shares, which is the methodology used by the Bureau of Labor Statistics to compute the CPI.

An alternative SCOLI is the democratic index, Π^D, which uses unitary weights on household COLIs instead of expenditure weights:

$$\Pi^D = \frac{1}{H} \sum_{h=1}^{H} \pi_h = \frac{1}{\sum_{i=1}^{N} \frac{1}{n_i}} \sum_{i=1}^{N} \frac{1}{n_i} \pi_i. \tag{3}$$

Here, individual COLIs are weighted by the reciprocal of the number of household members. The avoidance of expenditure weights is the great advantage of the democratic index (see, e.g., Ley 2005).

Both the plutocratic and democratic indices lack a solid welfare economic foundation. Pollak (1981) offers a SCOLI which is explicitly grounded in a welfare economic problem. Define the direct welfare function $\omega = W(u_1, ..., u_N)$ to give the level of social welfare ω corresponding to a vector of utilities $u_1, ..., u_N$. Define the indirect welfare function, Ω, to be $\Omega(\mathbf{p}, x_1, ..., x_N, \mathbf{z}_1, ..., \mathbf{z}_N) = W(V(\mathbf{p}, x_1, \mathbf{z}_1), ..., V(\mathbf{p}, x_N, \mathbf{z}_N))$, which is a function of prices, expenditures and demographics. Define the indirect social cost function $M(\mathbf{p}, \omega, \mathbf{z}_1, ..., \mathbf{z}_N)$ as the minimum total (across households) expenditure required to attain the level of social welfare ω for a population with characteristics $\mathbf{z}_1, ..., \mathbf{z}_N$ facing prices $\mathbf{p}$.

Pollak's proposal for a SCOLI is

$$\Pi^M = \frac{M(\mathbf{p}_1, \omega, \mathbf{z}_1, ..., \mathbf{z}_N)}{M(\mathbf{p}_0, \omega, \mathbf{z}_1, ..., \mathbf{z}_N)}$$

where ω equals initial welfare, new welfare, or some other welfare level. Here, the numerator is equal to the minimum total expenditure across all households required to get a welfare level of ω when facing prices $\mathbf{p}_1$, and the denominator is the minimum total expenditure when facing prices $\mathbf{p}_0$.

Pollak's is a very elegant solution to the aggregation problem. However, even with the welfare function in hand, this procedure requires an optimization step in which the investigator determines the optimal distribution of expenditure in each price regime. This can be hard, and Pollak's proposal has not been widely adopted.[2]

2.3   The Common-Scaling SCOLI and First-Order Approximation

We propose a social cost-of-living index, Π^*, which is similar in spirit to the individual COLI defined by (1). The individual COLI equates the utility of scaled expenditure when facing $\mathbf{p}_1$ to the utility of expenditure when facing $\mathbf{p}_0$. We define the common-scaling social cost-of-living index (CS-SCOLI), Π^*, as the single scaling of *all* expenditures that equates welfare at the two different price vectors. We solve

$$W\left(V(\mathbf{p}_0, x_1, \mathbf{z}_1), ..., V(\mathbf{p}_0, x_N, \mathbf{z}_N)\right) = W\left(V(\mathbf{p}_1, \Pi^* x_1, \mathbf{z}_1), ..., V(\mathbf{p}_1, \Pi^* x_N, \mathbf{z}_N)\right) \qquad (4)$$

for Π^*. Just as a person's cost-of-living index is the scaling to her expenditure that holds her utility constant over a price change, the CS-SCOLI is the scaling to *everyone's* expenditure that holds social welfare constant over a price change.

---

[2]We note that Slesnick (2001) implements Pollak's SCOLI. However, apart from Slesnick, we know of no other investigators using Pollak's SCOLI.

If preferences are identical across individuals (even those with different demographic characteristics), then the optimisation problem is much easier. In this case, the optimal distribution of expenditure is one characterised by equality.

A first-order approximation of Π^* may be obtained by approximating W around $\mathbf{p}_0$ and $x_1, ..., x_N$. This yields

$$(\mathbf{p}_1 - \mathbf{p}_0)' \sum_{i=1}^{N} \frac{\partial W(\cdot)}{\partial V(\mathbf{p}_0, x_i, \mathbf{z}_i)} \frac{\partial V(\mathbf{p}_0, x_i, \mathbf{z}_i)}{\partial \mathbf{p}} + (\Pi^* - 1) \sum_{i=1}^{N} \frac{\partial W(\cdot)}{\partial V(\mathbf{p}_0, x_i, \mathbf{z}_i)} \frac{\partial V(\mathbf{p}_0, x_i, \mathbf{z}_i)}{\partial x_i} x_i = 0$$

where $W(\cdot)$ denotes $W(V(\mathbf{p}, x_1, \mathbf{z}_1), ..., V(\mathbf{p}, x_N, \mathbf{z}_N))$. Rearranging terms and substituting the logarithmic form of Roy's Identity, $\mathbf{w}_i = -\frac{\partial V(\mathbf{p}_0, x_i, \mathbf{z}_i)}{\partial \ln \mathbf{p}} / \frac{\partial V(\mathbf{p}_0, x_i, \mathbf{z}_i)}{\partial \ln x_i}$, gives

$$\Pi^* \approx 1 + \left(\frac{\mathbf{p}_1 - \mathbf{p}_0}{\mathbf{p}_0} \right)' \frac{\sum_{i=1}^{N} \frac{\partial W(\cdot)}{\partial V(\mathbf{p}_0, x_i, \mathbf{z}_i)} \frac{\partial V(\mathbf{p}_0, x_i, \mathbf{z}_i)}{\partial x_i} x_i \mathbf{w}_i}{\sum_{i=1}^{N} \frac{\partial W(\cdot)}{\partial V(\mathbf{p}_0, x_i, \mathbf{z}_i)} \frac{\partial V(\mathbf{p}_0, x_i, \mathbf{z}_i)}{\partial x_i}},$$

which expresses the approximation in terms of weighted averages of the budget share vectors, $\mathbf{w}_i$. We may rewrite this expression as

$$\Pi^* \approx 1 + \left(\frac{\mathbf{p}_1 - \mathbf{p}_0}{\mathbf{p}_0} \right)' \sum_{i=1}^{N} \phi_i \mathbf{w}_i \tag{5}$$

where

$$\phi_i = \frac{\frac{\partial W(\cdot)}{\partial V(\mathbf{p}_0, x_i, \mathbf{z}_i)} \frac{\partial V(\mathbf{p}_0, x_i, \mathbf{z}_i)}{\partial x_i} x_i}{\sum_{i=1}^{N} \frac{\partial W(\cdot)}{\partial V(\mathbf{p}_0, x_i, \mathbf{z}_i)} \frac{\partial V(\mathbf{p}_0, x_i, \mathbf{z}_i)}{\partial x_i} x_i} \tag{6}$$

is the 'welfare-weight' of an individual. Note that the expenditure-share weighted proportional price change,

$$1 + \left(\frac{\mathbf{p}_1 - \mathbf{p}_0}{\mathbf{p}_0} \right)' \mathbf{w}_i,$$

is a first-order approximation to an individuals' cost-of-living index. Thus (5) can be interpreted as the welfare-weighted average of first-order approximations to individual cost-of-living indices.

If W is weakly concave in u_i and V is weakly concave in x_i, then ϕ_i must be weakly concave in x_i. A polar case thus obtains if W is linear in V and V is linear in x_i, resulting in no social aversion to expenditure inequality. In this case, ϕ_i is equal to the individual's share of expenditure, $x_i / \sum_{i=1}^{N} x_i$. Below, we show that this is equivalent in the first order to the Plutocratic index. However, this obtains if and only if social welfare is not averse to inequality of utility and marginal utility is constant, neither of which are appealing conditions.

Another case of interest obtains if the product $\frac{\partial W(\cdot)}{\partial V(\mathbf{p}_0, x_i, \mathbf{z}_i)} \frac{\partial V(\mathbf{p}_0, x_i, \mathbf{z}_i)}{\partial x_i}$ is equal to $1/x_i$, so that $\phi_i = 1$ for all i. This case yields an index which (as discussed further below) is equivalent in the first order to the democratic index, except that the identity weights apply to individuals rather than households. Other welfare weights yield other SCOLIs.

Equation (5) gives a first-order approximation to a common-scaling SCOLI for any choice of social welfare function and indirect utility functions. It nests as special cases both the first-order approximation to the plutocratic index and an index similar in spirit to the democratic index. Given the welfare weights, the common-scaling SCOLI is easily calculated from standard data sources: it is the weighted average of proportional price changes, where the weights are themselves weighted averages of individual household expenditure shares.

2.4   Second-Order Approximation

Equation (4) defines the CS-SCOLI (Π^*) implicitly. A first-order approximation to (4) is of course linear in Π^*, and thus easily solved. Higher order approximations, however, will be nonlinear in Π^*. It is easier therefore, to place a restriction on the social welfare function which allows us to derive an explicit expression for Π^*. This can then be approximated at higher orders.

We proceed by suppressing the $\mathbf{p}_1$ argument of welfare, which can be done without loss of generality, and by suppressing the $\mathbf{z}_1, ..., \mathbf{z}_N$ arguments of welfare, which may be done with an equivalent-expenditure function (defined below). Then, it turns out that all we

8

need for an explicit solution for Π^* is that the indirect welfare function Ω is homothetic in it's (equivalent) expenditure arguments. As we discuss below, homotheticity of the indirect welfare function does not imply homotheticity of the indirect utility function.

We may define the CS-SCOLI in terms of the indirect welfare function Ω, solving

$$\Omega(\mathbf{p}_0, x_1, ..., x_N, \mathbf{z}_1, ..., \mathbf{z}_N) = \Omega(\mathbf{p}_1, \Pi^* x_1, ..., \Pi^* x_N, \mathbf{z}_1, ..., \mathbf{z}_N) \tag{7}$$

for Π^*. Because Ω is *welfarist*, it is invariant to changes which leave the utility vector unchanged. Thus, we may substitute in (1) on the left-hand side and rewrite the implicit equation for Π^* as

$$\Omega(\mathbf{p}_1, \pi_1 x_1, ..., \pi_N x_N, \mathbf{z}_1, ..., \mathbf{z}_N) = \Omega(\mathbf{p}_1, \Pi^* x_1, ..., \Pi^* x_N, \mathbf{z}_1, ..., \mathbf{z}_N), \tag{8}$$

which is evaluated entirely at $\mathbf{p}_1$.

In (8), π_i depends on $\mathbf{p}_1, \mathbf{p}_0, x_i$ and z_i, and Π^* depends on $\mathbf{p}_1, \mathbf{p}_0$ and x_i and z_i, $i = 1, ..., N$. The dependence of π_i and Π^* on their arguments is suppressed. Here, Ω does not vary with $\mathbf{p}_1$ because any changes to $\mathbf{p}_1$ are exactly offset by changes in π_i that hold the utility vector constant, as follows from the definition of π_i. We can therefore suppress $\mathbf{p}_1$ on both sides of this definition, and rewrite it as

$$\overline{\Omega}(\pi_1 x_1, ..., \pi_N x_N, \mathbf{z}_1, ..., \mathbf{z}_N) = \overline{\Omega}(\Pi^* x_1, ..., \Pi^* x_N, \mathbf{z}_1, ..., \mathbf{z}_N), \tag{9}$$

where $\overline{\Omega}$ suppresses direct dependence on $\mathbf{p}_1$.

Indirect welfare functions are not necessarily symmetric over expenditure because households vary in their characteristics. However, symmetry of direct welfare over utility implies symmetry of indirect welfare over *equivalent-expenditure*, the expenditure level which equates utility across household types. Given a reference household type $\overline{\mathbf{z}}$, the equivalent-expenditure

of person i, x_i^e, solves

$$V(\mathbf{p}, x_i^e, \overline{\mathbf{z}}) = V(\mathbf{p}, x_i, \mathbf{z}_i).$$

Consequently, two people with the same level of equivalent-expenditure have the same level of utility. If equivalent-expenditure is proportional to expenditure, we say that "equivalence-scale exactness" (ESE) holds, and in this case we may write

$$x_i^e = \frac{x_i}{\Delta(\mathbf{p}, \mathbf{z}_i)},$$

where Δ is the "equivalence scale" giving the ratio of expenditure needs across different types of households and people (see Blackorby and Donaldson 1993).

Assume now that equivalence-scale exactness holds and that the indirect welfare function Ω is symmetric and homothetic over equivalent-expenditure. This pair of assumptions ensures that Ω is homothetic over the expenditure vector because ESE implies that scaling the expenditure distribution by some factor scales the equivalent-expenditure distribution by the same factor. Homotheticity of Ω over expenditure makes social indifference curves independent of the units of measure of expenditure, and implies that Ω is ordinally equivalent to a function Ω_0 which is homogeneous of degree 1 in x:

$$\Omega(\mathbf{p}, x_1, ..., x_N, \mathbf{z}_1, ..., \mathbf{z}_N) = f\left(\Omega_0(\mathbf{p}, x_1, ..., x_N, \mathbf{z}_1, ..., \mathbf{z}_N)\right)$$

where f is monotone and

$$\lambda\Omega_0(\mathbf{p}, x_1, ..., x_N, \mathbf{z}_1, ..., \mathbf{z}_N) = \Omega_0(\mathbf{p}, \lambda x_1, ..., \lambda x_N, \mathbf{z}_1, ..., \mathbf{z}_N)$$

for any λ. The function Ω_0 is not a unique homogeneous representation of Ω, but all homogeneous representations of Ω are proportional to each other.

Now turning to the CS-SCOLI, the assumption of homotheticity of indirect welfare yields a solution for Π^*. Denoting the homogeneous representation of $\overline{\Omega}$ as $\overline{\Omega}_0$, we have

$$\overline{\Omega}_0(\pi_1 x_1, ..., \pi_N x_N, \mathbf{z}_1, ..., \mathbf{z}_N) = \overline{\Omega}_0(\Pi^* x_1, ..., \Pi^* x_N, \mathbf{z}_1, ..., \mathbf{z}_N),$$

and substituting in equivalent-expenditure yields

$$\overline{\Omega}_0(\pi_1 x_1^e, ..., \pi_N x_N^e, \overline{\mathbf{z}}, ..., \overline{\mathbf{z}}) = \overline{\Omega}_0(\Pi^* x_1^e, ..., \Pi^* x_N^e, \overline{\mathbf{z}}, ..., \overline{\mathbf{z}}).$$

Letting $\widetilde{\Omega}$ denote $\overline{\Omega}_0$ with dependence on $\mathbf{z}_1, ..., \mathbf{z}_N$ suppressed, we may solve for Π^* as

$$\Pi^* = \frac{\widetilde{\Omega}\left(\pi_1 x_1^e, ..., \pi_N x_N^e\right)}{\widetilde{\Omega}\left(x_1^e, ..., x_N^e\right)}. \tag{10}$$

Here, the CS-SCOLI is the ratio of indirect welfare given 'inflated expenditure' to indirect welfare given expenditure. If $\mathbf{p}_1$ is 'higher' ('lower') than $\mathbf{p}_0$, then the π_i are bigger (smaller) than 1, and the numerator is bigger (smaller) than the denominator, implying that Π^* is bigger (smaller) than one.

Homotheticity of Ω is a real restriction that some combinations of direct social welfare functions and indirect utility functions will not satisfy. Nevertheless, it admits interesting cases. For example, it is satisfied if the direct social welfare function is utilitarian and the indirect utility is PIGL (which includes as cases both quasi-homothetic and Almost Ideal demands.) Moreover, in applied work on inequality, it is almost always assumed that Ω is homothetic. For example, all relative inequality indices (such as the Gini coefficient, S-Gini indices, Atkinson indices and Generalised Entropy indices) correspond to homothetic indirect social welfare functions.

2.5 Special Cases

To implement the CS-SCOLI, a particularly simple indirect welfare function is the Atkinson (or 'mean-of-order-r') family, which may be written as

$$
\widetilde{\Omega}\left(x_1^e, ..., x_N^e\right) = \left(\sum_{i=1}^{N} \left(x_i^e\right)^r\right)^{1/r}, \ r \neq 0, \ r \leq 1
$$

$$
= \exp\left(\sum_{i=1}^{N} \ln x_i^e\right), \ r = 0.
$$

Given this indirect welfare function, which depends on the parameter r governing inequality-aversion, we may write the CS-SCOLI, Π^*, as

$$
\Pi_r^* = \left(\frac{\sum_{i=1}^{N} \left(\pi_i x_i^e\right)^r}{\sum_{i=1}^{N} \left(x_i^e\right)^r}\right)^{1/r}, r \neq 0, \ r \leq 1 \tag{11}
$$

$$
= \exp\left(\sum_{i=1}^{N} \ln \pi_i\right), \ r = 0
$$

where the subscript denotes the value of the inequality aversion parameter.

In the case where $r = 1$ there is no inequality aversion, and the CS-SCOLI takes the form

$$
\Pi_1^* = \frac{1}{\sum_{i=1}^{N} x_i^e} \sum_{i=1}^{N} x_i^e \pi_i. \tag{12}
$$

If we rewrite this inequality-neutral CS-SCOLI in terms of household COLIs, the connection with the plutocratic SCOLI, Π^P, becomes clear. We have

$$
\Pi_1^* = \frac{1}{\sum_{i=1}^{N} x_i^e} \sum_{i=1}^{N} x_i^e \pi_i = \frac{1}{\sum_{h=1}^{H} n_h x_h^e} \sum_{h=1}^{H} n_h x_h^e \pi_h,
$$

and if $n_h x_h^e = x_h$, this becomes the plutocratic SCOLI given by (2). The restriction that $n_h x_h^e = x_h$ is equivalent to the restriction that the equivalence scale Δ is equal to n_h so that equivalent-expenditure equals per-capita household expenditure. Thus, $\Pi^P = \Pi_r^*$ if $r = 1$ and $x_h^e = x_h/n_h$.

In the case where $r = 0$, the CS-SCOLI is given by

$$\Pi_0^* = \prod_{i=1}^{N} \pi_i^{1/N}, \tag{13}$$

the geometric mean of individual COLIs. This CS-SCOLI is dual to an inequality-averse indirect welfare function, and it down-weights extreme values of individual COLIs. If we rewrite this inequality-averse CS-SCOLI in logs and in terms of household COLIs, the connection with the democratic SCOLI, Π^D, becomes clear. We have

$$\ln \Pi_0^* = \frac{1}{N} \sum_{i=1}^{N} \ln \pi_i = \frac{1}{\sum_{h=1}^{H} n_h} \sum_{h=1}^{H} n_h \ln \pi_h,$$

and if $n_h = 1$ for all h, then the log of this CS-SCOLI is the unweighted average of individual log COLIs. Thus the Π^D and Π_r^* with $r = 0$ differ only in that: the democratic SCOLI is an arithmetic mean and the CS-SCOLI is a geometric mean; and the democratic SCOLI uses unitary weights for households and the CS-SCOLI uses unitary weights for individuals.

The democratic SCOLI and the CS-SCOLI with $r = 0$ share the feature that they are functions of household COLIs that do not depend on household expenditure levels. One might argue that this is key the feature of the democratic SCOLI that makes it a desirable alternative to the plutocratic SCOLI. Thus, it is reasonable to ask if there are indirect welfare functions lying outside the Π_r^* class whose implied CS-SCOLIs do not depend on household expenditure levels and are different from Π_0^*, the CS-SCOLI with $r = 0$. The proposition below establishes that there are no such alternative indirect welfare functions. That is, Π_0^* is the only CS-SCOLI which does not depend on household expenditure levels.

Proposition 1 *The CS-SCOLI defined by (10) is independent of equivalent-expenditure if and only if it is the geometric mean of individual cost-of-living indices.*

Proof. See Appendix B. ∎

3 Estimation

Consumer demand data are typically household-level data describing expenditure on commodities linked to aggregate data on the prices of those commodities. Because consumer demand micro-data are expensive to collect and process, price data are typically available before consumption data. It is very common to use household expenditure data from past years to evaluate the cost-of-living given current prices. In this section, we show how to use household expenditure data collected in the past and current commodity price data to construct second-order approximations of all the SCOLIs discussed in the previous subsection (for an example of approximation of social welfare change, see Banks, Blundell and Lewbel 1996).

Since the Plutocratic and Democratic SCOLIs are similar to cases of the CS-SCOLI, we begin by approximating the Π_r^* for an arbitrary value of r. Using the fact that $\pi_h = C(\mathbf{p}, u_h, z_h)/x_h$, we may write Π_r^* in terms of household-level variables and as a function of household cost functions as follows:

$$\Pi_r^*(\mathbf{p}) = \frac{\left(\sum_{i=1}^N (\pi_i x_i^e)^r\right)^{1/r}}{\left(\sum_{i=1}^N (x_i^e)^r\right)^{1/r}} = \frac{\left(\sum_{h=1}^H n_h (\pi_h x_h^e)^r\right)^{1/r}}{\left(\sum_{h=1}^H n_h (x_h^e)^r\right)^{1/r}} \tag{14}$$

$$= \frac{\left(\sum_{h=1}^H n_h \left(x_h^e \frac{C(\mathbf{p}, u_h, z_h)}{x_h}\right)^r\right)^{1/r}}{\left(\sum_{h=1}^H n_h (x_h^e)^r\right)^{1/r}}, \tag{15}$$

where $x_h^e = x_h/\delta_h$ and $\delta_h = \Delta(\mathbf{p}_0, z_h)$ (so that ESE is a maintained assumption).

We construct second-order Taylor approximations of this expression at a new price vector $\mathbf{p}_1$ by expanding around $\mathbf{p}_0$. The only way prices enter Π_r^* is through the cost function C. The first-order part of the approximation uses Sheppard's lemma, which equates the first derivative of cost with demands, and the second-order part uses the Slutsky theorem, which links Marshallian price and expenditure derivatives with Hicksian price derivatives. In the following lemma, we show a second-order approximation of $\Pi_r^*(\mathbf{p}_1)$.

Lemma 2 *The second-order Taylor approximation of $\Pi_r^*\left(\mathbf{p}_1\right)$ about $\mathbf{p}_0$ may be expressed as*

$$\Pi_r^*\left(\mathbf{p}_1\right) \approx 1 + d\mathbf{p}'\overline{\mathbf{w}}^r + \frac{1}{2}d\mathbf{p}'\left[\overline{\mathbf{\Gamma}}^r + r\overline{\mathbf{w}\mathbf{w}'}^r + (1-r)\,\overline{\mathbf{w}}^r\overline{\mathbf{w}}^{r\prime} - \widetilde{\mathbf{W}}^r\right]d\mathbf{p}$$

where $d\mathbf{p} \equiv \left(\mathbf{p}_1 - \mathbf{p}_0\right)/\mathbf{p}_0$,

$$
\begin{aligned}
\overline{\mathbf{w}}^r &\equiv \frac{1}{\left(\sum_{h=1}^{H} n_h \left(x_h^e\right)^r\right)} \sum_{h=1}^{H} n_h \left(x_h^e\right)^r \mathbf{w}_h, \\
\overline{\mathbf{w}\mathbf{w}'}^r &\equiv \left(\frac{1}{\left(\sum_{h=1}^{H} n_h \left(x_h^e\right)^r\right)} \sum_{h=1}^{H} n_h \left(x_h^e\right)^r \mathbf{w}_h \mathbf{w}_h'\right),
\end{aligned}
\tag{16}
$$

$\widetilde{\mathbf{W}}^r$ is a diagonal matrix $\overline{\mathbf{w}}^r$ on the main diagonal, and

$$\overline{\mathbf{\Gamma}}^r \equiv \frac{1}{\left(\sum_{h=1}^{H} n_h \left(x_h^e\right)^r\right)} \sum_{h=1}^{H_{\mathbf{p}_0}} n_h \left(x_h^e\right)^r \mathbf{\Gamma}_h,$$

where

$$\mathbf{\Gamma}_h \equiv \nabla_{\ln \mathbf{p}}\mathbf{w}(\mathbf{p}_0, x_h, \mathbf{z}_h) + \nabla_{\ln \mathbf{x}}\mathbf{w}(\mathbf{p}_0, x_h, \mathbf{z}_h)\mathbf{w}(\mathbf{p}_0, x_h, \mathbf{z}_h)'. \tag{17}$$

Proof. See Appendix B. ∎

The lemma shows a second-order approximation for $\Pi_r^*\left(\mathbf{p}_1\right)$ which depends on weighted averages, $\overline{\mathbf{w}}^r$, of expenditure-share vectors, $\mathbf{w}_h$, and weighted averages, $\overline{\mathbf{\Gamma}}^r$, of compensated semi-elasticity matrices, $\mathbf{\Gamma}_h$. We consider in turn: (1) first-order approximations; (2) second-order approximations in the absence of unobserved preference heterogeneity; and (3) second-order approximations in the presence of unobserved preference heterogeneity.

The first-order approximation is given by

$$\Pi_r^*\left(\mathbf{p}_1\right) \approx 1 + d\mathbf{p}'\overline{\mathbf{w}}^r, \tag{18}$$

which is equivalent to (5) with welfare weights $\phi_i = \frac{1}{\left(\sum_{i=1}^{N}\left(x_i^e\right)^r\right)} \sum_{i=1}^{N}\left(x_i^e\right)^r$. The expenditure-share vectors, $\mathbf{w}_h$, are observed for all households, so the weighted average, $\overline{\mathbf{w}}^r$, may always be calculated directly from the data (without an estimation step). Further, $\overline{\mathbf{w}}^r$ may be estimated consistently regardless of presence of unobserved preference heterogeneity.

In the absence of unobserved preference heterogeneity, the weighted average of compensated semi-elasticity matrices, $\overline{\mathbf{\Gamma}}^r$, is estimable via (weighted) average derivative estimation. Thus, in the absence of unobserved preference heterogeneity, all of the terms in the second-order approximation of $\Pi_r^*\left(\mathbf{p}_1\right)$ given in Lemma 2 may be estimated consistently.

With unobserved preference heterogeneity, things are more complicated. In the presence of unobserved preference heterogeneity which is independent of observables $(\mathbf{p}, x, \mathbf{z})$, the population-level weighted average of $\mathbf{w}_h$ may still be estimated consistently via the sample weighted average of $\mathbf{w}_h$. Thus, $\overline{\mathbf{w}}^r$, $\widetilde{\mathbf{W}}^r$, $\overline{\mathbf{w}}^r\overline{\mathbf{w}}^{r\prime}$, and $\overline{\mathbf{w}\mathbf{w}}^r$ may all be estimated consistently. However, even with this limited form of preference heterogeneity, Lewbel (2001) shows that the sample weighted-average of $\mathbf{\Gamma}_h$ is not a consistent estimator of $\overline{\mathbf{\Gamma}}^r$. This is because, although the unobserved preference heterogeneity is assumed independent of observables $(\mathbf{p}, x, \mathbf{z})$, if, at a particular $(\mathbf{p}, x, \mathbf{z})$, *both* the derivatives and levels depend on unobserved preference parameters (heterogeneity), then the product $\nabla_{\ln \mathbf{x}}\mathbf{w}(\mathbf{p}_0, x_h, \mathbf{z}_h)\mathbf{w}(\mathbf{p}_0, x_h, \mathbf{z}_h)'$ may have a nonzero expectation.

We propose a new estimator of $\overline{\mathbf{\Gamma}}^r$ which exploits the symmetry of the Slutsky matrix and circumvents Lewbel's problem.

Proposition 3 *Under the maintained assumption of Slutsky symmetry,*

$$\overline{\mathbf{\Gamma}}^r \equiv \frac{1}{\left(\sum_{h=1}^{H} n_h\left(x_h^e\right)^r\right)} \sum_{h=1}^{H_{\mathbf{P}0}} n_h\left(x_h^e\right)^r \mathbf{\Gamma}_h^s, \tag{19}$$

with

$$\mathbf{\Gamma}_h^s \equiv \mathbf{\Gamma}(\mathbf{p}_0, x_h, \mathbf{z}_h) = \frac{1}{2}\left(\nabla_{\ln \mathbf{p}}\mathbf{w}(\mathbf{p}_0, x_h, \mathbf{z}_h) + \nabla_{\ln \mathbf{p}}\mathbf{w}(\mathbf{p}_0, x_h, \mathbf{z}_h)' + \nabla_{\ln x}\left(\mathbf{w}(\mathbf{p}_0, x_h, \mathbf{z}_h)\mathbf{w}(\mathbf{p}_0, x_h, \mathbf{z}_h)'\right)\right).$$

Proof. See Appendix B. ∎

The intuition here is that although the sample weighted-average of $\mathbf{\Gamma}_h$ does not yield a consistent estimate of $\overline{\mathbf{\Gamma}}^r$, the sample weighted-average of $\mathbf{\Gamma}_h + \mathbf{\Gamma}'_h$ is a consistent estimate of $\overline{\mathbf{\Gamma}}^r + \left(\overline{\mathbf{\Gamma}}^r\right)'$, which given symmetry is equal to $2\overline{\mathbf{\Gamma}}^r$. The matrix $\mathbf{\Gamma}_h$ contains a derivative $(\nabla_{\ln x}\mathbf{w}(\cdot))$ multiplied by a level $(\mathbf{w}(\cdot)')$, which may be polluted by covariance between the derivative and the level. In contrast, the analogous part of $\mathbf{\Gamma}_h^s = \frac{1}{2}\left(\mathbf{\Gamma}_h + \mathbf{\Gamma}'_h\right)$ contains a derivative of a product $(\nabla_{\ln x}\left(\mathbf{w}(\cdot)\mathbf{w}(\cdot)'\right))$. The covariance within this product does not pollute the estimate of the average derivative of the product. This is because, if unobserved preference heterogeneity parameters (error terms) are independent of $(\mathbf{p}, x, \mathbf{z})$, then given $(\mathbf{p}, x, \mathbf{z})$, the derivative of the average is the average of the derivative. Thus, we can consistently estimate the expectation of $\mathbf{\Gamma}_h^s$ locally at a particular $(\mathbf{p}, x, \mathbf{z})$, and aggregate across $(\mathbf{p}, x, \mathbf{z})$ to obtain a consistent estimate of the population-level weighted average derivative, $\overline{\mathbf{\Gamma}}^r$. In an environment with independent unobserved preference heterogeneity, we may estimate $\overline{\mathbf{\Gamma}}^r$ under Slutsky symmetry with any standard average derivative estimator, suitably modified to include the weights $n_h \left(x_h^e\right)^r$, and use the estimate to compute the second-order approximation given by Lemma 2 and Proposition 3.

3.1 Cases of Interest

Consider the inequality-neutral case where $r = 1$. In this case, we have

$$\Pi_1^* \approx 1 + d\mathbf{p}'\overline{\mathbf{w}}^1 + \frac{1}{2}d\mathbf{p}'\left[\overline{\mathbf{\Gamma}}^1 + \overline{\mathbf{w}\mathbf{w}'}^1 - \widetilde{\mathbf{W}}^1\right]d\mathbf{p} \tag{21}$$

where sample averages and average derivatives are weighted by equivalent expenditure. Here, the cross-product of weighted-average budget shares drops out of the approximation. If $x_h^e = x_n/n_h$, so that equivalent-expenditure is equal to per-capita household expenditure,

the plutocratic SCOLI results.

The democratic SCOLI may be written in terms of household cost functions as

$$\Pi^D = \frac{1}{H} \sum_{h=1}^{H} \pi_h = \frac{1}{H} \sum_{h=1}^{H} \frac{C(p, u, z)}{x_h},$$

which just amountings to reweighting Π^P. Thus, the second-order approximation of the democratic SCOLI is

$$\Pi^D \approx 1 + d\mathbf{p}'\overline{\mathbf{w}}^D + \frac{1}{2}d\mathbf{p}'\left[\overline{\mathbf{\Gamma}}^D + \overline{\mathbf{w}\mathbf{w}'}^D - \widetilde{\mathbf{W}}^D\right]d\mathbf{p} \tag{22}$$

where $\overline{\mathbf{w}}^D$, $\overline{\mathbf{\Gamma}}^D$, $\overline{\mathbf{w}\mathbf{w}'}^D$ and $\widetilde{\mathbf{W}}^D$ are unweighted (across households) averages. The only difference between (21) and (22) is in the weighting of the averages.

For the CS-SCOLI with $r = 0$, which is inequality-averse, the second-order approximation is given by

$$\Pi_0^* \approx 1 + d\mathbf{p}'\overline{\mathbf{w}}^0 + \frac{1}{2}d\mathbf{p}'\left[\overline{\mathbf{\Gamma}}^0 + \overline{\mathbf{w}}^0\overline{\mathbf{w}}^{0\prime} - \widetilde{\mathbf{W}}^0\right]d\mathbf{p}. \tag{23}$$

Here, the averages are weighted by the number of members in each household, and the weighted-average of cross-products of budget shares is replaced by a cross-product of weighted average budget shares.

4 Illustration

The approximate SCOLIs described above all employ weighted averages of expenditure shares and weighted average derivatives of expenditures shares. Weighted sample means and weighted average derivatives both converge at $\sqrt{H}$ where H is the number of households (observations) if weights are strictly positive and bounded. Given that ESE implies x_h^e is positive if x_h is positive and that both x_h and n_h are positive by definition, if we add the

18

restriction that both x_h and n_h are bounded, the condition on weights is satisfied.

The estimation of weighted average derivatives may be implemented by various empirical strategies. For example, Deaton and Ng (1998) use Hardle and Stoker's (1989) estimator which does not use estimates of derivatives of demands for any particular observations, but rather recovers the average derivative by multiplying the derivative of the density function with the level of demand. We use a more direct approach: we use a high dimensional nonparametric kernel estimator to generate an estimate of the matrix of derivatives $\mathbf{\Gamma}_h^s$, defined in equation (20), for all $h = 1, ..., H$. Then, we compute $\overline{\mathbf{\Gamma}}^r$, defined in equation (19), as the weighted average of $\mathbf{\Gamma}_h^s$. Although our estimation strategy differs from Deaton and Ng, the average derivative is characterized by fast convergence because we average over the H slowly-converging kernel estimates of $\mathbf{\Gamma}_h^s$. Thus, both weighted averages and weighted average derivatives converge at $\sqrt{H}$, where H is the number of households that face $\mathbf{p}_0$.

We use household-level microdata on expenditures from the American Consumer Expenditure Surveys (CES), 1980 to 1998, and aggregate commodity price data from 1980 to 2000, both of which are publicly available from the Bureau of Labor Statistics (BLS). These are the data which underlie the Consumer Price Index produced by the BLS. The CPI is the weighted average of commodity price changes, where the weights are equal to aggregate commodity expenditure shares, which may be interpreted as household expenditure-weighted household expenditure shares. Thus, the CPI is a first-order approximation to the Plutocratic index, Π^P, as described above.

We estimate our model using household expenditures in 19 distinct price regimes representing annual commodity prices for 9 goods for each year 1980 to 1998. The CES microdata are available at the monthly and quarterly level, but since our commodity price data are annual over calendar years, we use only households for which a full year of expenditure is available, with the full year starting in December, January or February. Since the rental flows from owned accommodation are difficult to impute and commodity prices are available only for urban residents, we use only rental-tenure urban residents of the continental USA.

19

There remain 4705 households in our restricted sample, with approximately 300 observations in each year from 1980 to 1998. Following Harris and Sabelhaus (2000), we reweight all household data to reflect these sample restrictions. These weights are used in the constructing sample weighted averages and weighted average derivatives, but not in the kernel estimation step. Summary statistics for the sample are presented in Appendix A.

The nine commodities are: food at home; food out; rent; household furnishing and operation; clothing; motor fuel; public transportation; alcohol; and tobacco products. These commodities account for approximately 3/4 of household consumption for households in the sample. We account for two household demographic characteristics: the number of household members; and the age of the head of the household. To compute equivalent-expenditure, we use an equivalence scale equal to the square root of the number of household members, which is the "standard" equivalence scale in the measurement of inequality. Results are essentially identical if we instead use equivalence scales based on the ratios of official US poverty lines for different household sizes.

Table 1 gives estimated first-order approximations of various SCOLIs. We do not provide standard errors in the table because the variance in the estimates due to the variance of the estimates of $\overline{\boldsymbol{\Gamma}}^r$, $\overline{\mathbf{w}\mathbf{w}'}^r$ and $\overline{\mathbf{w}}^r$ is very small (bootstrapped standard errors are less than 0.05 percentage points for all estimates shown).[3] We present illustrative results for 2 periods: 1988 to 1998 and 1999 to 2000. The former period is chosen because the BLS used 1982-4 expenditure weights for calculating the CPI over that entire period. (Since the late 1990s, the BLS has updated the weights used in the CPI about every 2 years.) In our example, we use 1983 expenditure weights for that period. The latter period is chosen because although most prices were fairly stable over 1999 to 2000, the price of motor fuel rose by 30% over this year. Since motor fuel represents a comparatively large expenditure share for the bottom half of households, we may expect the distributional weights to matter over such a price change. We use 1998 expenditure weights to assess this price change.

---

[3]As noted by Ley (2000, 2005), the variance induced by by the variance of $\overline{\boldsymbol{\Gamma}}^r$, $\overline{\mathbf{w}\mathbf{w}'}^r$ and $\overline{\mathbf{w}}^r$ is likely dwarfed by the variance induced by measurement error in proportional price changes, $d\mathbf{p}$.

On the left-hand side of Table 1 we present the plutocratic and democratic indices, and on the right-hand side, we present 3 CS-SCOLIs. Recall that the Π_1^* and Π_0^* indices are similar to the plutocratic and democratic indices, respectively, but are motivated from an explicit welfare foundation. The Π_{-1}^* index is not analogous to any commonly used SCOLI, but may also be motivated from a welfare foundation. The first-order approximations aggregate proportional price changes with weighted averages of expenditure shares. For the Π_0^* index, the weights are uniform across individuals. For the Π_1^* index, the weights are directly proportional to equivalent-expenditure and for the Π_{-1}^* index, the weights are inversely proportional to equivalent-expenditure.

Table 1: Estimated SCOLIs, percentage changes

expenditure		first-order				
base		Pluto	Demo	CS-SCOLIs		
		Π^P	Π^D	Π_1^*	Π_0^*	Π_{-1}^*
1988-1998	1983	32.5	33.5	32.5	33.5	34.4
1999-2000	1998	4.5	4.6	4.5	4.6	4.7

Over the long period 1988 to 1998, the plutocratic index rose by 32.5%.[4] However, because price changes favoured rich households over poor households during this period, the democratic index which up-weights the experience of poor households, rose by 33.5%. The estimated first-order approximations of the CS-SCOLIs which are similar to these indices are identical to the first decimal place (but differ beyond that). The CS-SCOLI which puts yet more weight on the experience of poor households, Π_{-1}^*, is larger still, with an estimated increase in the social cost-of-living of 34.4%. Thus, over this 10-year period, we see that different plausible weighting structures in the common-scaling SCOLI yield different pictures of the path of inflation. In particular, the index which emphasizes the experience of poor households shows 2 percentage points more inflation than that which emphasizes

---

[4]Over 1988 to 1998 and 1999 to 2000, the CPI rose by 35.6% and 3.3%, respectively. We do not expect the CPI and our Plutocratic index to be numerically identical because the CPI is computed from a much larger and finer set of commodities.

the experience of rich households. These results are consistent with other studies showing variation in the cost-of-living across income classes (see, e.g., Pendakur 2002, Ley 2002, Chiru 2005a,b).

In considering the one-year price change for 1999 to 2000, we use expenditure weights from 1998 because the BLS has recently announced that they will update expenditure weights every 2 years with a 2 year lag. Here, we see that the large increase in the relative price of motor fuel had a noticeable distributional effect. The plutocratic and democratic SCOLIs are 4.5% and 4.6%, respectively, due to the fact that motor fuel is a necessity whose price increases affects poor households more than rich households. The welfare-derived CS-SCOLIs illustrate the same story, with the Π_1^* showing a 4.5% increase in the social cost-of-living and the Π_{-1}^* index showing a 4.7% increase in the social cost-of-living.

Table 2 presents estimates of first- and second-order approximations of the CS-SCOLIs for the same years. Here, we may illustrate the importance of accounting for substitution effects in the assessment of the social cost-of-living. Since the second-order term in the approximation is an average of a quadratic form in the Slutsky matrix of each household, accounting for substitution effects must (weakly) reduce the estimated SCOLI if all households are rational. For all the comparisons below, this is the case.

Table 2: Estimated SCOLIs, percentage changes

	expenditure	first-order			second-order		
	base	Π_1^*	Π_0^*	Π_{-1}^*	Π_1^*	Π_0^*	Π_{-1}^*
1988-1998	1983	32.5	33.5	34.4	32.0	32.6	33.3
1999-2000	1998	4.5	4.6	4.7	4.5	4.5	4.6

For the period 1988 to 1998, accounting for substitution effects reduces our estimate of the increase in inequality-neutral CS-SCOLI, Π_1^*, index from 32.5% to 32.0%, a difference of 0.5 percentage points. Turning to the two inequality-averse CS-SCOLIs, Π_0^* and Π_{-1}^*, the reduction is 0.9 and 1.1 percentage points, respectively. This magnitude for substitution effects is plausible given the high level of commodity aggregation in our illustration, though

somewhat smaller than the magnitudes identified in the Boskin Report (1996).

It is natural to expect that substitution effects will matter more if expenditure weights are updated infrequently or with long lags. This is because if expenditure weights are updated continuously and instantly, a fine sequence of first-order approximations will capture the behavioral responses that the substitution effects 'predict' (see Vartia 1983). Since the BLS has substantially reduced delays and increased the frequency of expenditure updates, it may be important to assess the size of second-order effects over short periods. Turning to the one-year price change from 1999 to 2000 which uses 1998 expenditure weights, we still see substitution effects of noticeable magnitude. During this period, the price of motor fuel rose by 30%, which is large enough in principle to induce changes in behavior to reduce the cost of the price change. The bottom row of Table 2 suggests that this was the case. We see that the estimated first-order approximations of the Π_0^* and Π_{-1}^* CS-SCOLIs are 0.1 percentage points higher than the estimated second-order approximations. Although this effect is small, such effects may 'add up' over long periods of time because substitution effects always have the same (negative) sign.

5 Conclusion

For an individual, the change in the cost-of-living is the scaling of expenditure required to hold utility constant over a price change. Because preferences and resources differ across people, for any price change, there is heterogeneity across individuals in their cost-of-living changes. Thus, a social cost-of-living approach to the measurement of price change faces a formidable aggegation problem. Whose cost-of-living should we be measuring?

The common-scaling social cost-of-living index (CS-SCOLI) developed in this paper answers the following question. What single scaling to everyone's expenditure would hold social welfare constant across a price change? This index has social welfare foundations, and allows the investigator to easily choose the weight placed on rich and poor households. It is easy to

implement, and we have provided methods for estimating both first- and second-order approximations to the index. The latter capture substitution effects. Our estimation methods allow for unobserved preference heterogeneity.

Finally, the CS-SCOLI has as special cases objects that are either identical or very similar to all the commonly used social cost-of-living indices, and in particular, the plutocratic and democratic SCOLIs. This is important. In our framework there is a social welfare function and equivalence scale which lead to the CPI. Thus the CPI is given an explicit social welfare foundation. Moreover, an investigator that finds the social welfare function and equivalence scale corresponding to the CPI unpalatable can easily generate a SCOLI more to her tastes.

6   Appendix A: Summary Statistics

Table A1 gives summary statistics for the data we use in our analysis.

Table A1: The Data		Min	Max	Mean	Std Dev
expenditure shares	food-in	0	0.85	0.26	0.13
	food-out	0	0.63	0.08	0.07
	rent	0	0.94	0.41	0.15
	hh furn/equip	0	0.45	0.04	0.05
	clothing	0	0.41	0.06	0.05
	motor fuel	0	0.43	0.07	0.06
	public trans	0	0.39	0.09	0.04
	alcohol	0	0.54	0.03	0.04
	tobacco	0	0.26	0.03	0.04
log-expenditure		6.66	10.76	9.05	0.55
log household size		0	2.56	0.65	0.59
age of head (less 40)		-24	24	2.9	11

7 Appendix B: Proofs

7.1 Proof of Proposition 1

Proof. The CS-SCOLI given by (10) may be written as a function of equivalent-expenditures, x_i^e, and individual COLIs, π_i, as follows:

$$\Pi^* \left(\pi_1, ..., \pi_N, x_1^e, ..., x_N^e \right) = \frac{\widetilde{\Omega} \left(\pi_1 x_1^e, ..., \pi_N x_N^e \right)}{\widetilde{\Omega} \left(x_1^e, ..., x_N^e \right)}$$

If Π^* is independent of x_i^e, $i = 1, ..., N$, then we may write it as a function, $\overline{\Pi}^*$, depending only on π_i:

$$\overline{\Pi}^* \left(\pi_1, ..., \pi_N \right) = \frac{\widetilde{\Omega} \left(\pi_1 x_1^e, ..., \pi_N x_N^e \right)}{\widetilde{\Omega} \left(x_1^e, ..., x_N^e \right)}.$$

This may be rewritten as

$$\widetilde{\Omega} \left(\pi_1 x_1^e, ..., \pi_N x_N^e \right) = \overline{\Pi}^* \left(\pi_1, ..., \pi_N \right) \widetilde{\Omega} \left(x_1^e, ..., x_N^e \right), \tag{24}$$

which is a functional equation explored by Eichhorn (1978, equation 3.6.2). He shows that functional equations of the form (24) are satisfied if and only if $\widetilde{\Omega}$ is the weighted product function, given by

$$\widetilde{\Omega} \left(x_1^e, ..., x_N^e \right) = c \prod_{i=1}^{N} \left(x_i^e \right)^{k_i}$$

where c and k_i are constants. Since $\widetilde{\Omega}$ is symmetric, $k_i = k$, and since $\overline{\Pi}^*(1, ..., 1) = 1$, $k = 1/N$. Since $\widetilde{\Omega}$ is homothetic, we may set $c = 1$. Thus, the indirect welfare function is

$$\widetilde{\Omega} \left(x_1^e, ..., x_N^e \right) = \prod_{i=1}^{N} \left(x_i^e \right)^{1/N}$$

which is the CS-SCOLI given by equation (13) with $r = 0$. ∎

7.2 Proof of Lemma 2

Proof. The approximation is given by

$$\Pi_r^* (\mathbf{p}_1) \approx 1 + (\mathbf{p}_1 - \mathbf{p}_0)' \nabla_{\mathbf{p}} \Pi^r (\mathbf{p}_0) + \frac{1}{2} (\mathbf{p}_1 - \mathbf{p}_0)' \nabla_{\mathbf{pp}'} \Pi^r (\mathbf{p}_0) (\mathbf{p}_1 - \mathbf{p}_0)$$

which may be rewritten in terms of proportional changes $d\mathbf{p}$ as

$$\Pi_r^* (\mathbf{p}_1) \approx 1 + d\mathbf{p}' \widetilde{\mathbf{P}}_0 \nabla_{\mathbf{p}} \Pi^r (\mathbf{p}_0) + \frac{1}{2} d\mathbf{p}' \widetilde{\mathbf{P}}_0 \nabla_{\mathbf{pp}'} \Pi^r (\mathbf{p}_0) \widetilde{\mathbf{P}}_0 d\mathbf{p} \tag{25}$$

where $\widetilde{\mathbf{P}}_0$ is a diagonal matrix with $\mathbf{p}_0$ on the main diagonal. Application of the chain rule and Sheppard's lemma ($\nabla_{\mathbf{p}} C(\mathbf{p}_0, u, z) = \mathbf{q}(\mathbf{p}_0, u, z)$, the quantity vector), together with substitution of the duality condition $C(\mathbf{p}_0, u_h, z_h) = x_h$, and the definition of expenditure shares, $\mathbf{w}(\mathbf{p}, u, z) \equiv \widetilde{\mathbf{P}} \frac{\mathbf{q}(\mathbf{p}, u, z)}{x}$, implies that the first term may be expressed in terms of a weighted average of household expenditure shares:

$$
\begin{aligned}
d\mathbf{p}' \widetilde{\mathbf{P}} \nabla_{\mathbf{p}} \Pi^r (\mathbf{p}_0) &= d\mathbf{p}' \widetilde{\mathbf{P}}_0 \frac{\nabla_{\mathbf{p}} \left(\sum_{h=1}^H n_h \left(x_h^e \frac{C(\mathbf{p}_0, u_h, z_h)}{x_h} \right)^r \right)^{1/r}}{\left(\sum_{h=1}^H n_h (x_h^e)^r \right)^{1/r}} \\
&= d\mathbf{p}' \widetilde{\mathbf{P}}_0 \frac{\frac{1}{r} \left(\sum_{h=1}^H n_h \left(x_h^e \frac{C(\mathbf{p}_0, u_h, z_h)}{x_h} \right)^r \right)^{(1-r)/r} r \sum_{h=1}^H n_h \left(x_h^e \frac{C(\mathbf{p}_0, u_h, z_h)}{x_h} \right)^{r-1} x_h^e \frac{\mathbf{q}(\mathbf{p}_0, u_h, z_h)}{x_h}}{\left(\sum_{h=1}^H n_h (x_h^e)^r \right)^{1/r}} \\
&= d\mathbf{p}' \frac{\left(\sum_{h=1}^H n_h (x_h^e)^r \right)^{(1-r)/r} \sum_{h=1}^H n_h (x_h^e)^{r-1} x_h^e \widetilde{\mathbf{P}}_0 \frac{\mathbf{q}(\mathbf{p}_0, u_h, z_h)}{x_h}}{\left(\sum_{h=1}^H n_h (x_h^e)^r \right)^{1/r}} \\
&= d\mathbf{p}' \frac{\left(\sum_{h=1}^H n_h (x_h^e)^r \right)^{(1-r)/r} \sum_{h=1}^H n_h (x_h^e)^r \mathbf{w}(\mathbf{p}_0, u_h, z_h)}{\left(\sum_{h=1}^H n_h (x_h^e)^r \right)^{1/r}} \\
&= d\mathbf{p}' \overline{\mathbf{w}}^r
\end{aligned}
\tag{26}
$$

where

$$\overline{\mathbf{w}}^r \equiv \frac{1}{\left(\sum_{h=1}^{H} n_h \left(x_h^e\right)^r\right)} \sum_{h=1}^{H} n_h \left(x_h^e\right)^r \mathbf{w}_h.$$

Although much messier, the derivation of the second-order terms proceeds by the same process. Continued differentiation with use of the chain rule and re-application of Shephard's Lemma eventually yields

$$\frac{1}{2} \left(\mathbf{p}_1 - \mathbf{p}_0\right)' \nabla_{\mathbf{p}\mathbf{p}'} \Pi^r \left(\mathbf{p}_0\right) \left(\mathbf{p}_1 - \mathbf{p}_0\right) = \frac{1}{2} d\mathbf{p}' \left[\overline{\mathbf{\Gamma}}^r + r\overline{\mathbf{w}\mathbf{w}'}^r + (1 - r) \overline{\mathbf{w}}^r \overline{\mathbf{w}}^{r\prime} - \widetilde{\mathbf{W}}^r\right] d\mathbf{p} \quad (27)$$

where

$$\overline{\mathbf{\Gamma}}^r \equiv \frac{1}{\left(\sum_{h=1}^{H} n_h \left(x_h^e\right)^r\right)} \sum_{h=1}^{H_{\mathbf{p}_0}} n_h \left(x_h^e\right)^r \mathbf{\Gamma}_h,$$

with

$$\mathbf{\Gamma}_h \equiv \mathbf{\Gamma}(\mathbf{p}_0, x_h, \mathbf{z}_h) \equiv \nabla_{\ln \mathbf{p} \ln \mathbf{p}'} \ln C((\mathbf{p}_0, x_h, \mathbf{z}_h) = \nabla_{\ln \mathbf{p}} \mathbf{w}(\mathbf{p}_0, x_h, \mathbf{z}_h) + \nabla_{\ln \mathbf{x}} \mathbf{w}(\mathbf{p}_0, x_h, \mathbf{z}_h) \mathbf{w}(\mathbf{p}_0, x_h, \mathbf{z}_h)'.$$

The matrix-value function $\mathbf{\Gamma}(\mathbf{p}, x, \mathbf{z}) \equiv \nabla_{\ln \mathbf{p} \ln \mathbf{p}'} \ln C((\mathbf{p}, x, \mathbf{z})$ is the matrix of compensated semi-elasticities of the expenditure share equations $\mathbf{w}$, and $\mathbf{\Gamma}_h \equiv \mathbf{\Gamma}(\mathbf{p}_0, x_h, \mathbf{z}_h)$ gives the value of this function for each household facing $\mathbf{p}_0$. $\blacksquare$

7.3 Proof of Proposition 3

Proof. Given Slutsky symmetry, $\mathbf{\Gamma}(\mathbf{p}, u, \mathbf{z}) = \mathbf{\Gamma}(\mathbf{p}, u, \mathbf{z})'$, so that

$$\nabla_{\ln \mathbf{p}} \mathbf{w}(\mathbf{p}, x, \mathbf{z}) + \nabla_{\ln x} \mathbf{w}(\mathbf{p}, x, \mathbf{z}) \mathbf{w}(\mathbf{p}, x, \mathbf{z})' = \nabla_{\ln \mathbf{p}} \mathbf{w}(\mathbf{p}, x, \mathbf{z})' + \mathbf{w}(\mathbf{p}, x, \mathbf{z}) \nabla_{\ln x} \mathbf{w}(\mathbf{p}, x, \mathbf{z})',$$

which implies that

$$2\mathbf{\Gamma}(\mathbf{p},u,\mathbf{z}) = \nabla_{\ln\mathbf{p}}\mathbf{w}(\mathbf{p},x,\mathbf{z}) + \nabla_{\ln x}\mathbf{w}(\mathbf{p},x,\mathbf{z})\mathbf{w}(\mathbf{p},x,\mathbf{z})' + \nabla_{\ln\mathbf{p}}\mathbf{w}(\mathbf{p},x,\mathbf{z})' + \mathbf{w}(\mathbf{p},x,\mathbf{z})\nabla_{\ln x}\mathbf{w}(\mathbf{p},x,\mathbf{z})'.$$

Thus, we have

$$\mathbf{\Gamma}(\mathbf{p},u,\mathbf{z}) = \frac{1}{2}\left(\nabla_{\ln\mathbf{p}}\mathbf{w}(\mathbf{p},x,\mathbf{z}) + \nabla_{\ln x}\mathbf{w}(\mathbf{p},x,\mathbf{z})\mathbf{w}(\mathbf{p},x,\mathbf{z})' + \nabla_{\ln\mathbf{p}}\mathbf{w}(\mathbf{p},x,\mathbf{z})' + \mathbf{w}(\mathbf{p},x,\mathbf{z})\nabla_{\ln x}\mathbf{w}(\mathbf{p},x,\mathbf{z})'\right),$$

and $\mathbf{\Gamma}_h^s$, the symmetry-restricted estimated matrix of compensated semi-elasticities for a household h, may be written as in (20) ∎

References

[1] Banks, J., R. Blundell and A. Lewbel, 1996, "Tax Reform and Welfare Measurement: Do We Need Demand System Estimation," *Economic Journal,* 106:1227-1241.

[2] Blackorby, C. and D. Donaldson, 1993, "Adult Equivalence Scales and the Economic Implementation of Interpersonal Comparisons of Well-Being," *Social Choice and Welfare,* 10:335-61.

[3] Boskin et al., 1996, "Towards a More Accurate Measure of the Cost of Living,", Final report to the US Senate Finance Committee, Washington D.C., December.

[4] Boskin et al., 1998, "Consumer Prices, the Consumer Price Index, and the Cost of Living," *Journal of Economic Perspectives,* 12(1):3-26.

[5] Chiru, R., 2005a, "Is Inflation Higher for Seniors?" Statistics Canada Analytical Paper, 11-621-MIE – No. 027.

[6] Chiru, R., 2005b, "Does Inflation Vary with Income?" Statistics Canada Analytical Paper, 11-621-MIE – No. 030.

[7] Crawford, I. and Z. Smith, 2002, "Distributional Aspects of Inflation," The Institute for Fiscal Studies, Commentary 90.

[8] Deaton, A., 1998, "Getting Prices Right: What Should be Done?" *Journal of Economic Perspectives*, 12(1):37-46.

[9] Deaton, A., and S. Ng, 2002. "Parametric and Nonparametric Approaches to Price and Tax Reform " *Journal of the American Statistical Association*, 93(443): 900-909.

[10] Diewert, W., E., 1998, "Index Number Issues in the Consumer Price Index," *Journal of Economic Perspectives*, 12(1):47-58.

[11] Eichhorn, W., 1978, *Functional Equations in Economics*, Reading MA: Addison-Wesley.

[12] Hardle, W. and T. Stoker, 1989, "Investigating Smooth Multiple Regression by the Method of Average Derivatives," *Journal of the American Statistical Association*, 84:986-995.

[13] Harris, E. and J. Sabelhaus, 2000, "Consumer Expenditure Survey Family Level Extracts," National Bureau of Economic Research, http://www.nber.org/data/ces_cbo.html.

[14] Kokoski, M.F., 2000, "Alternative CPI Aggregations: Two Approaches," *BLS Monthly Labor Review*, 123:31–39.

[15] Lewbel, A., 2001, "Demand Systems With and Without Error Terms," *American Economic Review*, 91(3):611-618.

[16] Ley, E., 2005, "Whose Inflation? A Characterization of the CPI Plutocratic Gap," *Oxford Economic Papers* 57(4):634-646.

[17] Ley, E., 2002, "On Plutocratic and Democratic CPIs," *Economics Bulletin*, 4(3):1-5.

[18] Nicholson, J.L., 1975, "Whose Cost of Living?," *Journal of the Royal Statistical Society, Series A*, 138:540–2.

[19] Pendakur, K., 2002, "Taking Prices Seriously in the Measurement of Inequality," *Journal of Public Economics*, 86(1):47-69.

[20] Pollak, R., 1981, "The Social Cost-of-Living Index," *Journal of Public Economics*, 15:311–36.

[21] Prais, S., 1958, "Whose Cost of Living?" *Review of Economic Studies*, 26:126–34.

[22] Slesnick, D.T., 2001, *Consumption and Social Welfare: Living Standards and Their Distribution in the United States*, Cambridge, UK: Cambridge University Press.

[23] Vartia, Yrjo, 1983, "Efficient Methods of Measuring Welfare Change and Compensated Income in Terms of Ordinary Demand Functions", *Econometrica*, 51(1): pp. 79-98.