IS THE ELASTICITY OF INTERTEMPORAL SUBSTITUTION CONSTANT?

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Abstract
In all common models of inter-temporal allocation, the assumption of a constant elasticity of intertemporal substitution (EIS) imposes surprising limitations on within-period budget allocations. Consequently, the constant EIS assumption can be tested with demand data. In fact, the EIS is pinned down completely by the shape of Engel curves: if the EIS is constant then the EIS can be estimated without variation in the interest rate. That a price elasticity can be estimated without variation in the relevant price illustrates just how strong the constant EIS assumption is. The constant EIS assumption is rejected by demand data. (JEL: D91, E21, D12)

1. Introduction

Optimizing models of the intertemporal allocation of consumption are the workhorses of modern macroeconomics and public finance. Almost always, such models assume the power (or isoelastic) form for within-period utility (the "felicity function"). One reason is that, in combination with additivity over time, it yields homothetic (intertemporal) preferences, and this homotheticity is of considerable analytic convenience. For example, in a standard growth model, for a feasible steady state to be optimal requires that both consumption and marginal utility grow at a constant rate, and this requires intertemporal homotheticity.

Power felicity functions imply that the elasticity of intertemporal substitution (EIS) is constant: rich and poor agents are equally averse to proportional fluctuations in consumption. This is not an innocuous assumption. For example, as Chari and Kehoe (2006) point out, the well-known prescription of a zero tax on capital income follows because that quantitative general equilibrium models assume homothetic preferences over consumption at different dates (i.e. power felicity functions and a constant EIS).

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Standard Ramsey-type arguments then imply that intertemporal trade-offs should not be distorted. Thus, the policy prescription relies on the EIS being constant. More generally, the assumption of a constant EIS may have significant implications when evaluating the costs of business cycles or evaluating a policy change in a dynamic general equilibrium model with heterogeneous agents.

The goal of this paper is to highlight a surprisingly strong limitation that the assumption of a constant EIS imposes on the nature of the allocation of expenditure across goods *within a period*. In particular, the power form for preferences over total expenditure[1] that delivers a constant EIS requires that within-period preferences be from the PIGL/PIGLOG class. These preferences correspond to rank-2 demand systems. The rank of a demand system is the rank of the matrix of coefficients on income terms or, equivalently, the dimension of the space spanned by Engel curves (Lewbel 1991). In essence, to restrict the rank of a demand system is to place restrictions on the shapes that Engel curves for different goods can exhibit. For example, homothetic preferences over goods are of rank 1, and they imply that all Engel curves are linear and that goods are neither necessities nor luxuries. Increasing rank implies increasing flexibility. The rank-2 restriction on the shapes of Engel curves required for a constant-EIS is contradicted by a substantial body of empirical evidence on demands.

We show further that, if we assume a constant EIS and allow agents to consume both luxuries and necessities (so that within-period preferences are nonhomothetic), then the parameter of the power felicity function is pinned down completely by the shape of Engel curves and thus the EIS can be estimated without any variation in the intertemporal price. That a price elasticity can be estimated without variation in the relevant price illustrates just how strong the constant-EIS assumption is.

There is a large empirical literature that attempts to estimate the EIS using data on consumption growth. This turns out to be a difficult problem because such data are noisy, because we have limited variation in the intertemporal price (the interest rate), and because the relationship between the intertemporal price and consumption growth is mediated by liquidity constraints and uncertainty. With very few exceptions, this literature also assumes that the EIS is constant. The few papers that explore whether the EIS varies with the level of consumption (or wealth) seem to reject the constant-EIS hypothesis (Blundell, Browning, and Meghir 1994; Attanasio and Browning 1995; Atkeson and Ogaki 1996). However, these rejections tend not to be statistically strong, exactly for the reasons just listed. By contrast, the connection between intra- and intertemporal allocation developed in this paper provides for a much more powerful test of the constant-EIS assumption (and one that, at least implicitly, has already been carried out). Our test is more powerful because we have large amounts of household-level budget data and variation in total expenditure, compared with the limited amount of data available with variation in the intertemporal price. Our results support those of the papers that reject the constant-EIS assumption using consumption growth data.

1. Under expected utility, the power form is over total expenditure in each period. Under non-expected utility, as in Epstein and Zin (1989), the power form is over total expenditure in the current period and over the certainty equivalent in the next period.

It is also consistent with the results in Guvenen (2006), who shows that introducing heterogeneity in the EIS improves the fit of a calibrated macro model. Our result provides a theoretical explanation for these empirical findings: the EIS cannot be constant because the within-period budget allocations of rich and poor households differ in a complicated way.

In the next section we develop our main result in the familiar context of expected utility. In this context, the constant-EIS assumption corresponds to constant relative risk aversion (CRRA). In Section 3 we consider two important extensions. First, we show that our result holds for the more general recursive preferences proposed by Epstein and Zin (1989, 1991). These preferences relax the link between the EIS and risk aversion but maintain the constant-EIS assumption. Second, returning to an expected utility framework, we extend our analysis beyond the constant-EIS assumption to consider the more general class of HARA preferences where the EIS can vary in particular ways with wealth. Section 4 provides an empirical illustration, and Section 5 concludes.

2. Constant EIS with Expected Utility

Consider an agent who has an additive intertemporal utility function of the form

$$\sum \beta^t v(x_t),$$

where $v(x_t)$ is the "felicity" function that captures the utility derived from per-period "consumption", x_t. Of course, households consume many goods. We interpret $v(x_t)$ as the indirect utility function derived over total expenditure within the period and at within-period prices. This interpretation follows from two-stage budgeting, which holds because intertemporal preferences are additive. We should therefore write $v(x_t; p_t)$, where p_t is a vector of prices of different goods. This interpretation has been adopted by a number of papers that simultaneously examine inter- and intraperiod allocation (see e.g. Blundell, Browning, and Meghir 1994; Attanasio and Weber 1995). An alternative interpretation would treat $v(x_t)$ as a direct utility function defined over the composite consumption good, x_t. This relies either on (Hicks) composite commodity arguments (which require constant relative prices) or on the assumption of within-period homotheticity. We do not consider either approach to be credible. The assumption of constant relative prices is contrary to everyday experience and is particularly difficult to defend for an open economy (movements in gasoline prices are a good counterexample). More formally, that demand systems—including price responses—can be estimated on aggregated data (Deaton and Muellbauer 1980a is a classic example) is itself evidence of substantial variation in relative prices. The alternative assumption of within-period homotheticity implies that there are neither luxuries nor necessities; as Deaton (1992) notes, this "contradicts both common sense and more than a hundred years of empirical research." Homotheticity over goods is formally rejected not only by micro data but also in aggregate data (see e.g. Deaton and Muellbauer 1980a).

Throughout our analysis, we follow Browning (1985/2005) and define the elasticity of intertemporal substitution (EIS) as the derivative of log total expenditure with respect to the log of the intertemporal price (i.e., the interest rate), holding within-period relative prices and the discounted marginal utility of expenditure constant:

$$\text{EIS} = -\frac{\partial \log x_t}{\partial r_t} = -\frac{v_x(x_t; p_t)}{v_{xx}(x_t; p_t)\, x_t}. \tag{1}$$

The EIS is constant when it is independent of relative prices and of total expenditure. The reciprocal of the EIS is the elasticity of marginal utility with respect to consumption. In the expected utility context, this elasticity is the coefficient of relative risk aversion.

REMARK 1. The EIS is constant (i.e., independent of total within-period expenditure and relative prices) if and only if the felicity function has one of the following two forms:

$$v = a(p)\frac{x^{1-1/\theta}}{1-1/\theta} + b(p), \quad \theta \neq 1; \tag{2}$$

$$v = a(p)\log(x) + b(p), \quad \theta = 1. \tag{3}$$

These are the canonical forms of (respectively) PIGL and PIGLOG preferences (Muellbauer 1975, 1976).[2] Note that PIGL preferences (2) are homothetic if $\partial b/\partial p = 0$ and that PIGLOG preferences are homothetic if $\partial a/\partial p = 0$.

This result can be found in Muellbauer (1987) and Browning (1985/2005). Sufficiency follows directly from repeated differentiation of these utility functions with respect to total expenditure and substitution into equation (1). Necessity follows by taking expression (1), integrating twice, and then allowing the constants of integration to depend on prices. The point of Remark 1 is to note that the indirect utility functions resulting from the assumption of a constant EIS correspond to well-known demand systems. Neither Browning nor Muellbauer consider the potential value of this relationship for testing the constant-EIS assumption. Attanasio and Weber (1995) show that PIGLOG demands are also consistent with an EIS that depends on prices but not on total expenditure.

Under the constant-EIS assumption, the connection between intertemporal and intratemporal allocation is even stronger than this, as we now show.

PROPOSITION 1. *Suppose that either*

$$v = a(p)\frac{x^{1-1/\theta}}{1-1/\theta} + b(p) \quad \theta \neq 1, \;\; b_p(p) \neq 0, \tag{4}$$

2. PIGL preferences are those that can be represented by the utility function in equation (2). Any monotonic transformation of this utility function would generate the same within-period demands. PIGLOG preferences are those that can be represented by equation (3). PIGL/PIGLOG preferences are a subset of the generalized Gorman polar form (Gorman 1959) $u(x) = F[x/\alpha(p)] + \beta(p)$, where F is a monotone increasing function. PIGL/PIGLOG preferences have the convenient property of allowing exact nonlinear aggregation (over agents). See Muellbauer (1975, 1976) or Deaton and Muellbauer (1980a, 1980b).

or

$$v = a(p)\log x + b(p), \quad \theta = 1, \ a_p(p) \neq 0. \tag{5}$$

Then there is no transformation $F(v(x, p)) = \omega(x)$ such that: (i) $F(\cdot)$ is a monotonic single-argument function of v (in particular, independent of prices); (ii) $-\omega_x/(\omega_{xx}x) = \gamma$ (i.e., $-\omega_x/(\omega_{xx}x)$ is constant); and (iii) $\gamma \neq \theta$.

Proof. Consider first the PIGL case (4). The proof is by contradiction. Supposing that such an $F(\cdot)$ does exist, we proceed in two steps.

Step 1. Given (ii), using the chain rule in differentiating ω gives

$$-\frac{\omega_{xx}x}{\omega_x} = \frac{1}{\gamma} = -\frac{v_{xx}x}{v_x} - \frac{F_{vv}v_x x}{F_v} = \frac{1}{\theta} - k,$$

where θ is the power parameter (EIS) defined by the function v. Properties (i) and (ii) imply that k must be independent of prices and expenditure.

If $F(\cdot)$ is linear, then $F_{vv} = 0$ and thus $k = 0$. This means that $-\omega_{xx}x/\omega_x$ is constant but also that $\gamma = \theta$, contradicting property (iii). Thus $F(\cdot)$ must be nonlinear ($F_{vv} \neq 0$). Consequently, $F(\cdot)$ must satisfy

$$\frac{F_{vv}v_x x}{F_v} = k, \quad k \neq 0. \tag{6}$$

Step 2. Given (4), we know that $v_x = a(p)x^{-1/\theta}$. Substituting this into equation (6) yields

$$\frac{F_{vv}}{F_v} a(p)\, x^{1-1/\theta} = k. \tag{7}$$

Inverting (4) then gives

$$x = \left(\frac{v - b(p)}{a(p)}\left(1 - \frac{1}{\theta}\right)\right)^{\theta/(\theta-1)}.$$

Substituting back into equation (7), we obtain

$$\frac{F_{vv}}{F_v}(v - b(p))\left(1 - \frac{1}{\theta}\right) = k.$$

If we define Φ and Γ to be constants of integration, then this differential equation has the general solution

$$F(v, p) = \Phi \frac{1}{k\frac{1}{1-\frac{1}{\theta}} + 1}(v - b(p))^{k\frac{1}{1-1/\theta}+1} + \Gamma.$$

However, this $F(\cdot)$ is a function of p ($F_p \neq 0$), which contradicts (i). □

The argument for the PIGLOG case (5) follows the same steps.

Observe that linear transformations preserve the property that we are interested in (i.e., $-\omega_x/(\omega_{xx}x)$ is constant if $F(\cdot)$ is linear), but in that case $\gamma = \theta$. Also notice the importance of $b(p)$. By requiring that $b_p(p) \neq 0$, Proposition 1 is limited to the nonhomothetic case of PIGL and PIGLOG preferences. If $b_p(p) = 0$ then there are many transformations such that $-\omega_x/(\omega_{xx}x) = \gamma$ and $\gamma \neq \theta$. For example, any power function will do.

COROLLARY 1. *Assume:*

1. *intertemporal utility is additive in felicity;*
2. *the felicity function is of the form given in equation (2) or (3) (so that the EIS is constant); and*
3. *within-period preferences are nonhomothetic.*

Then the EIS is identified by the shape of the Engel curves.

Given that within-period preferences are ordinal, the reader may wonder why monotonic (price-independent) transformations of within-period preferences do not allow one to vary the curvature of the felicity function for any given within-period demands. This is what Proposition 1 rules out.

Another way to see this result is as follows. Let preferences over within-period allocations be represented by the function $u(x;\ p)$; let $F(u)$ be a single-argument, monotonically increasing transformation; and let the felicity function be $v(x,\ p) = F(u(x;\ p))$. Defining $H \equiv F^{-1}$, which is also an increasing monotonic transformation, we have

$$u\,(x; p) = H\left(a(p)\frac{x^{1-1/\theta}}{1-1/\theta} + b(p)\right), \quad \theta \neq 1;$$

$$u\,(x; p) = H\,(a(p)\log x + b\,(p)), \quad \theta = 1.$$

Then, by Roy's identity, the budget share of good j, w_j, is

$$w_j = -\frac{a_j}{a(p)(1-1/\theta)} - \frac{b_j}{a(p)}\,x^{1/\theta-1}, \quad \theta \neq 1; \tag{8}$$

$$w_j = -\frac{a_j}{a(p)}\log x - \frac{b_j}{a(p)}, \quad \theta = 1. \tag{9}$$

Here a_j and b_j are the derivatives of $a(p)$ and $b(p)$ with respect to the price of good j, p_j. The parameter θ determines the EIS (equation (2)) and the curvature of the Engel curve with respect to total within-period expenditure (equation (8)). If demand data identify a particular member of the PIGL/PIGLOG class, then they also identify the (constant) value of the EIS. One Engel curve is sufficient to identify the EIS.

Turning to a specific example, suppose u is a member of the PIGL family with curvature parameter $\eta \neq \theta$. This implies that u is some monotonic transformation $G(\cdot)$ of the canonical form of PIGL preferences:

$$u = G\left(\alpha\,(p)\,\frac{x^{1-1/\eta}}{1-1/\eta} + \beta\,(p)\right).$$

It follows that, if

$$v = a\,(p)\,\frac{x^{1-1/\theta}}{1-1/\theta} + b(p) = F(u),$$

then

$$v\,(x,\,p) = a\,(p)\,\frac{x^{1-1/\theta}}{1-1/\theta} + b\,(p) = M\left(\alpha\,(p)\,\frac{x^{1-1/\eta}}{1-1/\eta} + \beta\,(p)\right),$$

where the composite function $M(z) = F(G(z))$ is also an increasing monotonic transformation (independent of prices). Proposition 1 states that there is no such nonlinear transformation $M(z)$, while a linear $M(z)$ would imply $\eta = \theta$.

This example also makes clear the role of nonhomotheticity regarding within-period preferences. Homothetic PIGL preferences can be represented by $\alpha(p)x^{1-1/\gamma}/(1-1/\gamma)$, and the functional equation

$$a(p)\frac{x^{1-1/\theta}}{1-1/\theta} = M\left(\alpha(p)\frac{x^{1-1/\gamma}}{1-1/\gamma}\right)$$

has many solutions ($M(z)$ can be any power function.). That this result does not hold in the homothetic case makes intuitive sense: under PIGL with homothetic preferences, $b_j(p) = 0$ for all j, budget shares are flat for all values of θ and hence are not informative about the value of θ.

To see the connection to the specification studied by Attanasio and Weber (1995), suppose that within-period preferences are PIGLOG (with canonical form $\alpha(p)\log x + \beta(p)$), $G(z)$ is the exponential function, and $F(u)$ is a power function with parameter θ. Then

$$v = \frac{1}{1-1/\theta}\left(\exp\{\alpha(p)\ln x + \beta(p)\}\right)^{1-1/\theta}.$$

Then the EIS is defined by

$$\frac{1}{\text{EIS}} = \frac{1}{\theta}\alpha(p) - \alpha(p) + 1$$

and is independent of x but dependent on p. Note that here the parameter θ (and the EIS) is not identified by within-period demands. As Corollary 1 states, the EIS is identified by demands if and only if the EIS is independent of both expenditure and relative prices. With this specification, the EIS is independent both of expenditure and of relative prices if and only if within-period demand is homothetic—that is, $\alpha_j(p) = 0$ for all j.

There is a substantial literature (Ludvigson and Paxson 2001; Carroll 2001; Alan and Browning 2003; Attanasio and Low 2004) documenting the difficulties associated with estimating the EIS from data on consumption growth and the intertemporal price (Euler equation estimation). One response to these problems has been to move to structural estimation, as in Gourinchas and Parker (2002). Structural estimation brings its own difficulties, including that the environment in which agents operate must be completely—and correctly—specified. However, all of the cited papers *assume* that intertemporal preferences have the additive-over-time, power form. Thus it seems that this literature has been going to considerable effort to estimate a model parameter that, under the maintained assumptions of the model, could be estimated much more simply and easily.

However, the crucial point is that PIGL/PIGLOG (and hence a constant EIS) implies testable restrictions on the shapes of Engel curves. It has not been common,

in the literature, to test PIGL/PIGLOG preferences against more general alternatives.[3] On the other hand, PIGL/PIGLOG preferences are at most rank 2 (the homothetic form being rank 1), and the literature contains numerous tests of demand system rank. For example, Lewbel (1991), Banks, Blundell, and Lewbel (1997), and Donald (1997) provide nonparametric tests of rank, and Lewbel (2003) describes a parametric approach using a rational rank-4 demand system that nests rational rank-3 polynomial demands. The typical finding of this literature is that demands are at least of rank 3 and possibly of rank 4. In other words, there is a strong consensus that rank-2 demand systems (including PIGL/PIGLOG) provide an inadequate representation of intraperiod allocation, which implies that the EIS cannot be constant.

In terms of equations (8) and (9), a PIGL/PIGLOG specification of preferences can be tested in two ways. First, one can test whether the curvature of individual Engel curves is adequately captured by equations (8) and (9). That is, for each good, we can test equations (8) and (9) against more general specifications. Second, estimates of equations (8) and (9) for different goods should all give the same value of θ.

This approach to testing the constant-EIS assumption has considerable advantages over tests using consumption growth. The evidence from data on consumption growth is against a constant EIS (Blundell, Browning, and Meghir 1994; Attanasio and Browning 1995; Atkeson and Ogaki 1996). However, these tests face the same difficulties as Euler equation estimation. They use limited variation in the intertemporal price and noisy consumption growth data to assess not only how consumption growth responds to the intertemporal price (as in the usual exercise of Euler equation estimation) but also how that relationship varies with the level of consumption. It is perhaps not surprising, then, that the results of these tests (as in Blundell, Browning, and Meghir 1994) are often suggestive but not strongly statistically significant. In contrast, tests of the PIGL specification based on the curvature of Engel curves are powerful simply because we have so much household-level budget data and so much variation in total expenditure. This approach to testing the constant-EIS assumption does require that the EIS be homogeneous (as does a test based on consumption growth); however, because we have substantial household-level budget data, our test can be feasibly implemented on more homogeneous subgroups.

3. Extensions

3.1. Non-expected Utility

In the preceding analysis, we used an expected utility framework where the elasticity of intertemporal substitution is the reciprocal of the coefficient of relative risk aversion. Given the restrictiveness of this assumption, a number of researchers have moved to

3. More commonly, a particular member of this class (usually a parameterization of PIGLOG) is tested against a non-PIGL/PIGLOG parametric demand system that nests only that particular member of the class. See, for example, Banks, Blundell, and Lewbel (1997).

using more general preferences, following Epstein and Zin (1989, 1991). In this section, we show that the result tying the (constant) intertemporal elasticity to within-period demand holds also for these more general preferences.

Consider an agent maximizing utility at age t:

$$U_t = v^{-1}\left(\beta v\left(c_t\right) + (1-\beta)\, v\left(z_{t+1}\right)\right), \tag{10}$$

where z_{t+1} is the certainty equivalent of random future utility

$$z_{t+1} = g^{-1}(E[g\left(U_{t+1}\right)]), \tag{11}$$

c_t is "consumption" in period t (defined more carefully below), and $v(c_t)$ is the "felicity" function that captures the utility derived from period t consumption. Assume that $v' > 0$ and $v'' < 0$. Even in this non-expected utility setup, there is still additivity between within-period utility in period t and the value of the certainty equivalent in $t+1$.

In an expected utility setting, $v(\cdot) \equiv g(\cdot)$ and the model has the standard additive, intertemporal utility form

$$U_t = v^{-1}\left((1-\beta)\, E_t \sum_{j=0} \beta^j v(c_{t+j})\right). \tag{12}$$

Epstein and Zin (1991) specify that both the function governing intertemporal substitution and the function governing risk aversion are power functions:

$$U_t = \left[\beta \frac{c_t^{1-1/\theta}}{1-1/\theta} + (1-\beta)\frac{z_{t+1}^{1-1/\theta}}{1-1/\theta}\right]^{1/(1-1/\theta)}. \tag{13}$$

They choose this specification to give a constant EIS equal to θ.

Epstein and Zin (1991, p. 272) are explicit that they have in mind a single consumption good. However, an individual consumes more than one good and, in their empirical work, Epstein and Zin use total nominal expenditure deflated by a single price index. Using a single price index implies that intratemporal preferences are homothetic.

We can allow for more general intratemporal preferences in the spirit of the Epstein – Zin approach as follows. Let $u_t = u(x_t, p_t)$ be a representation of within-period preferences. The associated cost function $c(u_t, p_0)$ and reference price vector p_0 gives a monetary measure of within-period utility $c(u(x_t, p_t), p_0)$ for which we can calculate a certainty equivalent. This can be called "consumption". Thus:

$$U_t = v^{-1}\left(\beta v\left(c_t\right) + (1-\beta)\, v\left(z_{t+1}\right)\right), \tag{14}$$

$$\begin{aligned} z_{t+1} &= g^{-1}(E[g\left(U_{t+1}\right)]), \\ c_t &= c(u(x_t, p_t), p_0); \end{aligned} \tag{15}$$

so that

$$U_t = v^{-1}(\beta v(c(u(x_t, p_t), p_0)) + (1-\beta) v(z_{t+1})). \tag{16}$$

PROPOSITION 2. *The EIS (defined once again as* $-\partial \log x_t/\partial r_t$*) is independent of prices and total expenditures if and only if* $v(c(u_t(x_t, p_t), p_0))$ *takes the canonical PIGL/PIGLOG form:*

$$v(c(u(x_t, p_t), p_0)) = m(p_t)\, x_t^{1-1/\theta} + n(p_t), \quad \theta \neq 1, \tag{17}$$

$$v(c(u(x_t, p_t), p_0)) = m(p_t) \log x_t + n(p_t), \quad \theta = 1. \tag{18}$$

Proof. Just as in Remark 1, sufficiency follows directly from repeated differentiation of these utility functions with respect to total expenditure and substitution into the definition of the EIS. Necessity follows by taking the definition of the EIS (equation (1)), integrating twice, and then allowing the constants of integration to depend on prices. □

COROLLARY 2. *For the EIS to be constant with this multigood version of Epstein–Zin preferences, within-period preferences* $u(x_t, p_t)$ *must be PIGL/PIGLOG.*

Note that $v(\cdot)$ is monotonically increasing in c and also that given the reference price p_0, the cost function $c(u_t(x_t, p_t), p_0)$ is monotonically increasing in u. Thus, $v(c(u(x_t, p_t), p_0))$ is a monotonic transform of within-period preferences. Since $v(\cdot)$ is canonical PIGL/PIGLOG, it follows that within-period preferences must be in the PIGL/PIGLOG class.

The implication of this result is that demands can be used to test the constant-EIS assumption with this specification of recursive preferences. Thus the approach to testing the constant-EIS assumption proposed in this paper is more general than one that uses the expected utility framework.

PROPOSITION 3. *Assume:*

1. *preferences are given by (13);*
2. $v(c(u(x_t, p_t), p_0))$ *takes the canonical PIGL/PIGLOG form (equation (17)), so that the EIS is constant; and*
3. *within-period preferences* $u(x_t, p_t)$ *are not homothetic.*

Then the EIS is identified by the shape of the Engel curves.

Proof. Because within-period preferences are in the PIGL/PIGLOG class, a monotonic transformation $H(\cdot)$ of within-period preferences gives the canonical form. We focus on the PIGL case, but the steps in the argument are analogous for the PIGLOG case. We have

$$H(u_t(x_t, p_t)) = a(p_t) x_t^{1-1/\nu} + b(p_t).$$

Then

$$\begin{aligned} c_t &= c(u_t(x_t, p_t), p_0) \\ &= \left[\frac{H(u_t(x_t, p_t)) - b(p_0)}{a(p_0)}\right]^{1/(1-1/\nu)} \\ &= \left[\frac{a(p_t)x_t^{1-1/\nu} + b(p_t) - b(p_0)}{a(p_0)}\right]^{1/(1-1/\nu)}. \end{aligned}$$

Substituting this expression for consumption into the recursive utility framework gives

$$\begin{aligned} v(c_t) = v(c(u(x_t, p_t), p_0)) &= v\left(\left[\frac{a(p_t)x_t^{1-1/\nu} + b(p_t) - b(p_0)}{a(p_0)}\right]^{1/(1-1/\nu)}\right) \\ &= m(p_t)x_t^{1-1/\theta} + n(p_t), \end{aligned} \tag{19}$$

where ν is the parameter from within-period demands and θ is the intertemporal parameter. It follows from Proposition 1 that the only solution to this functional equation is $v(c_t) = c_t^{1-1/\theta}$ with $\theta = \nu$. □

As before, if within-period preferences are homothetic and so $b_j(p) = 0$ for all j, then the functional equation has many solutions and there is no restriction on the value of the EIS implied by within-period demand.

3.2. Money Metric Measures of Consumption

From the analysis of Section 3.1, it may seem that, if we define "consumption" as the money metric measure of utility as in equation (15) and if we define power felicity over this measure of consumption, then we can have a constant EIS of consumption (EIS $= -\partial \log c_t/\partial r_t$) for any underlying within-period demands. This is not correct. The key point is that the budget constraint is naturally defined in terms of total (nominal) expenditure x_t:

$$W_{t+1} = (1 + r_t)(W_t - x_t).$$

If preferences are homothetic, then (i) we can deflate nominal expenditure using a single price index and (ii) consumption is linear in total expenditure. This implies that the budget constraint is linear both in total expenditure and in consumption. With nonhomothetic preferences, to deflate nominal expenditure we need more than one price index, and consumption (defined as the money metric utility measure) must be nonlinear in total expenditure:

$$\begin{aligned} c_t = c(u(x_t, p_t), p_0) &= \left[\frac{(u(x_t, p_t) - b(p_0))(1 - 1/\theta)}{a(p_0)}\right]^{1/(1-1/\theta)} \\ &= \left[\left(\frac{a(p_t)}{a(p_0)}x_t^{1-1/\theta} + \frac{b(p_t) - b(p_0)}{a(p_0)}\left(1 - \frac{1}{\theta}\right)\right)\right]^{1/(1-1/\theta)}. \end{aligned}$$

Rearranging to give x as a function of c yields

$$x_t = f(c_t, p_t, p_0) = \left(\frac{a(p_0)}{a(p_t)} c_t^{1-1/\theta} - \frac{b(p_t) - b(p_0)}{a(p_t)} \left(1 - \frac{1}{\theta}\right) \right)^{1/(1-1/\theta)}. \quad (20)$$

We can now rewrite the budget constraint in terms of c:

$$f(c_{t+1}, p_{t+1}, p_0) = (1 + r)(W_t - f(c_t, p_t, p_0)).$$

The nonlinearity of (20) when $b_j(p) \neq 0$ means that the budget constraint must be nonlinear in consumption. This in turn means that, even with the power felicity function over consumption, the EIS is not independent of the level of consumption.

For example, in the expected utility framework, optimization with respect to c_t yields

$$c_t^{-1/\theta} - \lambda(1 + r) \frac{\partial f(c_t, p_t, p_0)}{\partial c_t} = 0;$$
$$-\frac{1}{\theta} \ln c_t = \ln \lambda + \ln(1 + r) + \ln \frac{\partial f(c_t, p_t, p_0)}{\partial c_t}.$$

This implicitly defines an EIS $(\partial \ln c_t / \partial r_t)$ that varies with c_t unless $\partial^2 f(c_t, p_t, p_0) / \partial c_t^2 = 0$, which holds only when $b_j(p) = 0$ for all j—that is, when demands are homothetic.

3.3. HARA Preferences

A natural extension to the assumption of a constant EIS is to allow the EIS to vary in particular ways with expenditure. For example, within the expected utility framework, the HARA class yields an EIS with the general form

$$\text{EIS} = -\frac{v_x}{v_{xx} x} = \theta + \gamma x^{-1}. \quad (21)$$

This class includes CRRA, which yields a constant EIS. Other members of the HARA class are quadratic, negative exponential, and translated power.[4] Within the HARA class, both increasing and decreasing relative risk aversion (or fluctuation aversion) are possible. Many important results in finance rest on the assumption of HARA preferences, including the two-fund separation theorems (see e.g., Eeckhoudt, Gollier, and Schlesigner 2005).[5]

4. Translated power is simply power defined over a translation of total expenditure:

$$u(x) = \frac{1}{\theta - 1} (\gamma + \theta x)^{1-1/\theta}.$$

The translation γ is sometimes interpreted as subsistence consumption.

5. Just as additivity plus the power functional form gives homothetic intertemporal preferences, additivity plus HARA preferences gives quasi-homothetic preferences over time periods or states (Pollak 1971). Intertemporal expansion paths (and Engel curves for periods or states) are linear but not through the origin. This linearity is crucial to aggregation results (just as in the intratemporal case).

Here we show the restrictions on within-period utility implied by this more general specification of preferences. In particular, analogously to Remark 1 and Proposition 1, we show that assuming intertemporal preferences are additive with utility functions of the HARA class implies demand functions that are of rank ≤ 3.

Solving equation (21) for v_x gives

$$v_x = A(p)(\gamma + \theta x)^{-1/\theta};$$

thus,

$$v = \frac{1}{\theta - 1} A(p)(\gamma + \theta x)^{1-1/\theta} + B(p).$$

Proposition 1 holds for HARA demands and hence its corollary does, too. To see this, define

$$Q = \frac{\gamma}{\theta} + x.$$

Then the definition of HARA preferences and the corresponding indirect utility function can be written as

$$-\frac{v_x}{v_{xx}} = \theta Q;$$

then

$$v = \frac{1}{\theta - 1} A(p)(\theta Q)^{1-1/\theta} + B(p),$$

and the proof to Proposition 1 given before can be applied directly (noting that $v_Q = v_x$ because $\partial Q/\partial x = 1$).

Therefore, under the HARA functional form, the EIS is still identified by the curvature of Engel curves (as long as within-period preferences are not homothetic), and the assumption of HARA preferences can be tested with budget data. Using Roy's identity, we have

$$x_j = -\frac{A_{p_j}\left(\frac{1}{\theta-1}\right)(\gamma + \theta x)^{1-1/\theta} + B_{p_j}}{A(p)(\gamma + \theta x)^{-1/\theta}}.$$

Thus share equations generated from an indirect utility function of the HARA class have the form

$$w_j = \frac{x_j}{x} = \frac{\theta}{1-\theta}\frac{A_{p_j}}{A(p)} + \frac{1}{1-\theta}\gamma\frac{A_{p_j}}{A(p)}x^{-1} + \frac{B_{p_j}}{A(p)}\frac{(\gamma + \theta x)^{1/\theta}}{x}. \tag{22}$$

Note that these demands have rank ≤ 3.[6]

6. Lewbel and Perraudin (1995) show that there is a correspondence between the rank of demands and the degree of fund separation; for example, two-fund separation is consistent with rank-2 demands. Note that this is a very different result from ours: Lewbel and Perraudin are referring to the rank of (general) preferences over states and are treating total consumption in each state as a single commodity.

In comparison to a specification with a constant EIS, this extra degree of rank means that there may be more scope for rationalizing HARA intertemporal preferences with within-period demand patterns. In practice, the two members of the HARA class that generate rank-3 demands are translated power and quadratic. The latter imply certainty equivalent behavior (no precautionary saving motive). Thus, if we wish to assume HARA preferences while ruling out quadratic utility (owing to the substantial empirical evidence of precautionary behaviour), then evidence of demands being rank 3 can be accommodated only by translated power utility. Translated power utility exhibits an EIS that rises with wealth: the rich are less averse to proportional fluctuations in consumption and are more inclined to move consumption across time to take advantage of the rate of return. Ogaki and Zhang (2001) show that the translated power utility function implies substantially different behavior from CRRA. Complete risk sharing is rejected using CRRA, but not when preferences are specified as translated power.[7]

Note also that the term in equation (22) that gives the extra degree of flexibility approaches zero as total within-period expenditure grows. As total expenditure grows, the rate of increase of the EIS decreases: translated power utility converges to the constant-EIS case, implying that the demands of the rich should be very close to rank 2. One way out of this might be to assume that the translation (γ) is determined by an external reference point as in models of "external habits" (Campbell and Cochrane 1999; Lettau and Uhlig 2000). Campbell and Cochrane show that such external habits can reconcile consumption behavior with stock market returns. Our results suggest that adding external habits to our consumption models may also help reconcile them with demand data. Such an extended model would imply that budget shares depend in testable ways on both total expenditure and the excess of total expenditure over the reference point (see equation (22)). If, instead of external habits, we were to introduce habits that were a function of past choices by the individual (as in Constantinides 1990), then this would break the intertemporal separability of consumers' choices and so Proposition 1 would not hold.

4. Empirical Illustration

As a brief illustration, we estimate Engel curves with micro data on expenditures from the 1997 and 1998 U.K. Family Expenditure Survey. To avoid (unobserved) within-period price variation, we focus on households in London and the South East (as in Banks, Blundell, and Lewbel 1997). We also focus on a restricted set of family types: couples with and without children.

7. In standard risk sharing models, the consumption growth of an agent (household or group) is related to aggregate consumption growth by that agent's relative risk aversion. Thus, under full insurance with CRRA, the consumption growth of each agent is equal to aggregate consumption growth and so the cross-section variance of consumption is unaffected by aggregate shocks. By contrast, full insurance with HARA preferences allows for changes in the cross-sectional variance of consumption in response to an aggregate shock.

TABLE 1. Estimates and tests of constant-EIS Engel curves.

Commodity	θ (EIS)	Het. Robust Confidence Interval	Test of Functional Form (Het. Consistent t-test)
Food	1.72	[1.31, 2.14]	−0.23
Clothing	1.43	[0.47, 2.40]	−1.31
Fuel and light	2.69	[1.50, 3.89]	−1.91
Leisure services	0.56	[0.47, 0.65]	−0.27
Test of equality	$\chi^2_3 = 43.74$		($p < 0.0001$)

As shown in equation (8), we can write the budget share equation for good j as

$$w_j = -\frac{a_j}{a(p)(1-1/\theta)} - \frac{b_j}{a(p)}\, x^{1/\theta-1}, \quad \theta \neq 1. \tag{23}$$

If the assumption of power utility is valid, then the curvature from θ should capture all the curvature in Engel curves and so the estimate of the parameter θ should be the same for any Engel curve. Table 1 reports, for four different goods, the estimate of θ (the EIS), a confidence interval for that estimate, the implied coefficient of relative risk aversion, and a test of equation (8). A system of four equations of the form of equation (23) was estimated by nonlinear least squares. Although we imposed the condition that θ must be constant across households, it is worth stressing that we did not restrict the degree of independent heterogeneity entering via the slope or intercept coefficients. The specific parametric alternative that we tested against is the general HARA share equation developed in Section 3.[8]

We begin, in the first row, with the food Engel curve. The estimate of θ is 1.7, which is higher than EIS estimates obtained from linearized Euler equation estimation: Attanasio and Weber (1995) estimate the EIS to be 0.67. However, estimates of the EIS in the literature vary substantially because of the difficulties with identification discussed earlier. For example, the EIS implied by the shape of the food Engel curve is within the wide range of (implied) EIS estimates that Gourinchas and Parker (2002) report ($\theta = 0.7$ to $\theta = 2.0$) based on their structural estimation strategy.

The second, third and fourth rows of Table 1 report similar estimates and tests for clothing, for fuel and light, and for leisure services. For clothing we obtain an estimate of θ similar to that derived from the food Engel curve. The estimate based on the fuel and light Engel curve is a bit higher, and both this and the clothing estimate are less precise than the estimate based on the food Engel curve. For leisure services, we estimate a θ that is substantially lower, around 0.5, and very precisely estimated.

Turning to testing, we first consider the restriction on the shape of each individual Engel curve. For none of the goods that we consider do we reject the functional form of equation (23) against a more flexible alternative (at a conventional 5% level of statistical significance).

8. We considered a number of other parametric alternatives, all of which gave similar results.

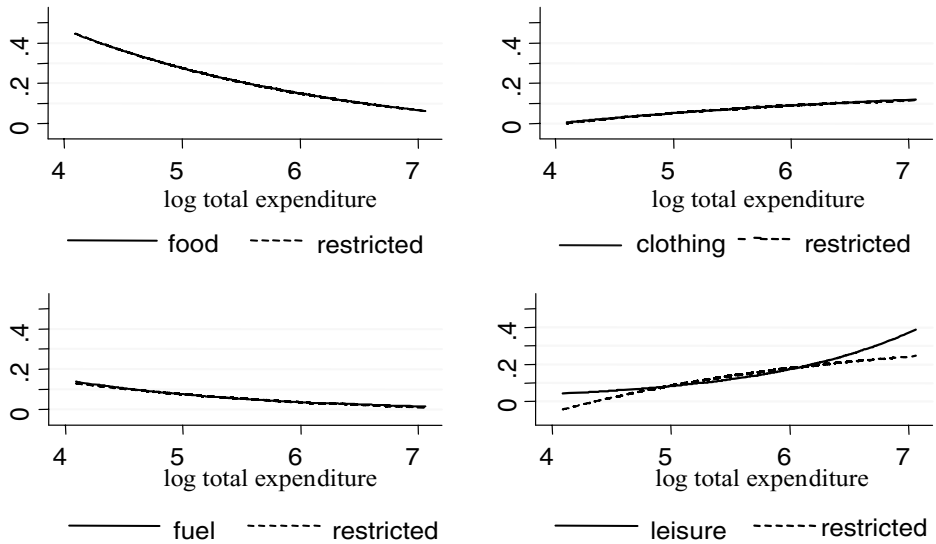

FIGURE 1. Restricted and unrestricted Engel curve estimates.

However, even an informal examination of the four estimates of θ and their confidence intervals reveals that restricting θ to be common across goods is strongly rejected by the data. A formal test gives a χ_3^2 test statistic of 43.7 and a p-value of less than 0.0001. Thus the data are incompatible with the constant-EIS hypothesis. This illustrates that the power of our demand-based test of the constant-EIS hypothesis derives from this restriction that θ be common across goods.

Our results are illustrated in Figure 1, which shows the fitted Engel curves when θ is unrestricted and when it is restricted to be common across goods. The optimal estimate of θ under this restriction is 1.72. The restricted and unrestricted Engel curves are quite similar for food, clothing, and fuel; however, they differ markedly for leisure. Thus, in this example it is the behavior of the leisure services share that cannot be reconciled with the constant-EIS hypothesis.

We emphasize again that this example is intended only as an illustration. The literature contains rejections of the hypothesis that demands are rank 2 based on parametric tests. However, these tests typically proceed by adding terms to polynomials in log expenditure—rather than testing against an alternative that corresponds to a particular class of utility function (e.g., translated power). The literature also contains rejections of rank 2 using nonparametric tests (Lewbel 1991). However, having demands that are rank 2 is necessary but not sufficient for having demands that are PIGL/PIGLOG. Thus a nonparametric test of rank is not strictly a direct test of the constant-EIS assumption but rather a test for a necessary condition. Having said this, the clear message from the literature is that demands cannot be adequately represented by rank 2. Finally, since PIGL preferences are necessary and sufficient for a constant EIS in a general recursive utility framework, these rejections of rank 2 are

also rejections of the constant-EIS assumption in our multigood version of Epstein and Zin's (1991) approach.

5. Discussion

This paper has shown that assumptions about the form of intertemporal preferences impose restrictions on within-period allocations: within-period demands must be in the PIGL/PIGLOG class if they are to be consistent with a constant EIS, whether the constant-EIS specification is in an expected utility CRRA framework or in an Epstein–Zin framework. This means that the assumption of a constant EIS requires within-period demands to be at most rank 2, a restriction that is typically rejected by demand data. Moreover, if within-period preferences are not homothetic (as is surely the case), if the EIS is constant, and if intertemporal preferences are either expected utility or Epstein–Zin, then the EIS can be estimated from a single Engel curve (without data on consumption growth or variation in the intertemporal price). This illustrates just how restrictive the constant-EIS assumption is. Finally, the connection between intra- and intertemporal allocation is not limited to the constant-EIS case. For example, the broader class of HARA (intertemporal) preferences are consistent with a particular form of intratemporal demands of at most rank 3.

More generally, our results reflect that all behavioral responses are governed by the curvature of the (indirect) utility function. Thus they are similar in spirit to Deaton (1974), who showed that additivity over goods implies a connection between (intratemporal) price and income elasticities, and to Deaton (1992) and Browning and Crossley (2000), who note that additivity over time *and* goods implies that the intertemporal substitution elasticities of particular goods are proportional to their income elasticities.

Our analysis shows that the assumption of a constant-EIS form for the felicity function is inconsistent with well-documented features of the micro data on intratemporal allocation. Our analysis supports the results of Blundell, Browning, and Meghir (1994) and of Attanasio and Browning (1995), who find evidence that the EIS is not constant. Our paper provides a further explanation for those results: the EIS cannot be constant because the within-period allocations of rich and poor households differ in a complicated way.

This leads to an obvious question: Just how wrong is the constant EIS assumption? In a statistical sense, our empirical results amount to a large rejection of the overidentifying restrictions implied by this functional form. It is more difficult to give an economic answer. The difficulty arises because with a general specification of the felicity function, we must use data on consumption growth and variation in the intertemporal price to characterize the EIS (and how the EIS varies with the level of consumption). As already discussed, this is a difficult task. However, we do know that relaxing the constant-EIS assumption can significantly change our answers to substantive questions. For example, Ogaki and Zhang (2001) show that it is important to allow for declining relative risk aversion (increasing EIS) when testing the full

risk-sharing hypothesis. Guvenen (2006) shows that allowing for heterogeneity in the EIS can improve the fit of a calibrated macro model.

The importance of our result is in refuting the belief that properties of intertemporal allocations can be independent of the properties of within-period allocation. This belief underpins the use of the constant-EIS assumption in much modern macroeconomics. We show that this assumption requires within-period allocations to take particular forms that are rejected by the evidence.

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