

Powerful t-Tests in the presence of nonclassical measurement error

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Abstract

This paper proposes a powerful alternative to the t-test in linear regressions when a regressor is mismeasured. We assume there is a second contaminated measurement of the regressor of interest. We allow the two measurement errors to be nonclassical in the sense that they may both be correlated with the true regressor, they may be correlated with each other, and we do not require any location normalizations on the measurement errors. We propose a new maximal t-statistic that is formed from the regression of the outcome onto a maximally weighted linear combination of the two measurements. Critical values of the test are easily computed via a multiplier bootstrap. In simulations, we show that this new test can be significantly more powerful than t-statistics based on OLS or IV estimates. Finally, we apply our test to the study of returns to education based on twins data from the U.S..

Keywords: linear regression, adaptive test, power of test, maximal combination of measurements, repeated measurements, multiplier bootstrap

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1 Introduction

The presence of measurement error in regressors causes the variance of OLS and IV estimators to be larger than those in the case without measurement error. As a result, t-tests of hypotheses about the regression coefficients may suffer from poor power. This problem can be particularly severe when the measurement errors are nonclassical, i.e. when they depend on the true level of the regressor, and/or when they depend on measurement errors of other measurements. In most economic applications in which the presence of measurement error is a first-order concern, neither of these two types of dependence can be ruled out. For example, earnings have been shown to contain measurement error so that high-earners under-report and low-earners over-report true earnings (Bound, Brown, and Mathiowetz (2001)). Therefore, measurement error must be nonclassical. In twins data such as that in Ashenfelter and Krueger (1994), the twins' reports on education have been argued to be mismeasured and nonclassical (e.g. Black, Berger, and Scott (2000), Hu and Sasaki (2017)). In this example, one would not want to rule out the possibility that the measurement errors of the twins are correlated because they are likely to possess correlated abilities to report correctly. In the literature on estimation of the skill production function (e.g. Cunha, Heckman, and Schennach (2010), Heckman, Pinto, and Savelyev (2013), Attanasio, Cattan, Fitzsimons, Meghir, and Rubio-Codina (2015), Attanasio, Meghir, and Nix (2017)), the measurements of skill production inputs are often elicited through the same data collection process so that different measurements of the same skill are likely to contain measurement errors that are correlated. More generally, any study that uses repeated measurements of a latent variable is likely to incur correlated measurement errors. Black, Berger, and Scott (2000) provide additional economic examples for which there is evidence for the presence of nonclassical measurement error.

In this paper, we propose to combine two measurements of the unobserved, true regressor so as to construct a new (“maximal”) t-statistic that we show, in simulations, leads to a more powerful test compared to standard t-tests. To describe the idea, consider the simple linear regression model

$$Y = \beta X^* + \varepsilon,$$

where Y is an outcome variable and X^* an unobserved regressor. Suppose we observe two measurements X and Z of X^* that are uncorrelated with the regression error ε . We want

to test the null hypothesis of no effect,

$$H_0: \beta = 0 \quad \text{vs.} \quad H_1: \beta \neq 0$$

using data on Y , X , and Z . If we knew a priori that the measurement errors in X and Z were classical and that the one in Z had a larger variance than that in X , then a t-test based on the OLS regression of Y on X would lead to a more powerful test than one based on the OLS regression of Y on Z . In that case, one might, for example, discard the noisier measurement Z and report only results from the regression of Y on X . The problem with this approach is that it crucially depends on the assumption of classical measurement error and on the knowledge of which measurement is less contaminated. Both of these are typically difficult, if not impossible, to justify.

The new “maximal” t-statistic proposed in this paper combines the two measurements in a data-driven fashion so as to make the t-statistic as large as possible. In the example above with classical and independent measurement errors, the maximal t-statistic will automatically put more weight on the precise measurement X relative to the noisy measurement Z without the user having to know a priori which one is more precise. However, for the new test to be valid we do not need to assume the measurement errors to be classical, i.e. they may depend on the true level of the regressor and the errors in the two measurements may depend on each other. In fact, we will see in the simulations that power gains relative to standard t-tests may be particularly large in the presence of nonclassical measurement error.

To develop the new maximal t-test we first introduce the class of IV estimands from a regression of Y onto an arbitrary linear combination of X and Z , using a possibly different linear combination as instrument. The OLS and IV estimands are special cases of this class, corresponding to particular weights in the linear combinations. The t-statistic derived from the estimator of the general estimand can then be optimized over the four weights, leading to our maximal t-statistic. By maximizing over the weights we test the null of no effect for all weights simultaneously, whereas the OLS and IV t-tests test the null only for specific choices of those weights.

We show that, with homoskedastic regression errors, there exists a closed-form solution for the maximal weights in the linear combinations, leading also to a closed-form expression of the maximal t-statistic. With more general heteroskedasticity-robust variance estimators, such closed-form expressions may not exist. It is immediate to see that

the maximal t-statistic is larger than standard OLS- or IV-based t-statistics. One might therefore wonder whether it is possible to construct a more powerful test based on this test statistic. We propose a multiplier bootstrap procedure for the calculation of critical values. Extensive Monte Carlo experiments confirm that the resulting maximal t-test is indeed very powerful. In all our simulation scenarios, the maximal t-test dominates all other OLS- and IV-based tests, sometimes by substantial margins, or dominates all but one, where this latter one performs similarly well.

We also show that it is not possible to improve [Black, Berger, and Scott \(2000\)](#)'s OLS and IV-based identification bounds on β by considering linear combinations of X and Z . The maximal t-test approach does therefore not lead to improved inference procedures in the more general hypothesis testing problem of

$$H_0: \beta = b \quad \text{vs.} \quad H_1: \beta \neq b$$

for some value $b \in \mathbb{R}$.

There is a large literature on identification and estimation of parametric and nonparametric models with measurement error. See, for example, the surveys by [Aigner, Hsiao, Kapteyn, and Wansbeek \(1984\)](#), [Bound, Brown, and Mathiowetz \(2001\)](#), [Chen, Hong, and Nekipelov \(2011\)](#), and [Schennach \(2013\)](#). Valid inference in parametric models is usually rather standard, once an identification argument and a consistent estimator have been provided. For example, the general approach by [Schennach \(2014\)](#) allows GMM-like moment conditions to depend on observables and unobservables, and shows how to convert them into equivalent moment conditions that depend only on observables. The testing of general hypotheses then proceeds by using standard GMM inference procedures.

However, we are not aware of any work trying to *improve* measurement error-robust inference methods by exploiting information in multiple measurements, which is the subject of this paper.

2 Maximal Combination of Measurements

In this section, we explore the possibility of maximizing the t-statistic by combining two measurements of the latent regressor. First, we consider testing the hypothesis of no effect and then the more general hypothesis for a pre-specified effect size. We show that, in the former problem, the t-statistic based on maximally weighted linear combinations is, in

general, larger than those based on the OLS and IV counterparts, but also that this result does not extend to the latter problem.

2.1 Testing the Hypothesis of No Effect

To explain the main idea of this section consider first the following simple regression model for a scalar outcome Y and a scalar regressor X^* satisfying

$$Y = \beta X^* + \varepsilon. \tag{1}$$

The outcome Y is observed, but X^* is not. Instead, we observe two measurements X and Z of X^* that may contain measurement errors (MEs) $U := X - X^*$ and $V := Z - X^*$. Defining ME to be additive is without loss of generality because we will neither impose any localization restriction on the ME, nor will we assume that it is independent of the true regressor X^* . In fact, we will not even restrict the dependence among the two MEs U and V .

Suppose we are interested in testing

$$H_0: \beta = 0 \quad \text{vs.} \quad H_1: \beta \neq 0. \tag{2}$$

The OLS estimand of a regression of Y on X and the IV estimand from a regression of Y on X using Z as instrument are

$$\beta_{OLS} := \frac{E[YX]}{E[X^2]} = \beta \frac{E[(X^*)^2] + E[X^*U]}{E[(X^*)^2] + 2E[X^*U] + E[U^2]}$$

$$\beta_{IV} := \frac{E[YZ]}{E[XZ]} = \beta \frac{E[(X^*)^2] + E[X^*V]}{E[(X^*)^2] + E[X^*U] + E[X^*V] + E[UV]}$$

Under the null, $\beta_{OLS} = \beta_{IV} = 0$, so the standard t-tests based on the OLS or IV estimators are asymptotically of correct level regardless of the dependence structure of the measurement system (i.e. the values of $E[X^*U]$, $E[X^*V]$, $E[UV]$). However, power of these two tests may be poor for two reasons: under any fixed and arbitrarily large alternative $\beta \neq 0$, at least one of the estimands β_{OLS} and β_{IV} is close to zero when (a) $E[X^*U]$ or $E[X^*V]$ is negative and large in absolute values or (b) one of $E[U^2]$, $E[X^*U]$, $E[X^*V]$, $E[UV]$ is large. Since the OLS and IV estimands depend on those quantities in different ways, the resulting t-tests may possess poor power under different data-generating processes.

We propose a new “maximal t-test” that is based on a class of estimands that contains OLS and IV as special cases. For any $a, b \in \mathbb{R}$, let $W(a, b) := aX + bZ$ be an arbitrarily weighted linear combination of the two measurements X and Z . For $\omega := (\omega_1, \omega_2, \omega_3, \omega_4)' \in \mathbb{R}^4$, consider the IV estimand from the regression of Y onto the linear combination $W(\omega_3, \omega_4)$ using a possibly different linear combination $W(\omega_1, \omega_2)$ as instrument,

$$\beta(\omega) := \frac{E[W(\omega_1, \omega_2)Y]}{E[W(\omega_1, \omega_2)W(\omega_3, \omega_4)]}. \quad (3)$$

The OLS estimand of regressing Y onto X (Z) corresponds to $\beta(1, 0, 1, 0)$ ($\beta(0, 1, 0, 1)$) and the IV estimand of regressing Y onto X (Z) using Z (X) as IV corresponds to $\beta(1, 0, 0, 1)$ ($\beta(0, 1, 1, 0)$).

The null implies that $\beta(\omega) = 0$ for all $\omega \in \mathbb{R}^4$ and, as long as $E[(X^*)^2] + E[X^*V] \neq 0$ and $E[(X^*)^2] + E[X^*U] \neq 0$, one can easily show that (2) is equivalent to

$$H_0: \sup_{\omega \in \mathbb{R}} |\beta(\omega)| = 0 \quad \text{vs.} \quad H_1: \sup_{\omega \in \mathbb{R}} |\beta(\omega)| \neq 0. \quad (4)$$

If the regression errors are homoskedastic, we standardize the estimand by

$$\sigma^2(\omega) := \frac{\sigma_\varepsilon^2(\omega) E[W(\omega_1, \omega_2)^2]}{(E[W(\omega_1, \omega_2)W(\omega_3, \omega_4)])^2},$$

where $\sigma_\varepsilon^2(\omega) := E[\varepsilon(\omega)^2]$ with $\varepsilon(\omega) := Y - \beta(\omega)W(\omega_3, \omega_4)$, is the residual variance. Given the reformulation of the null in (4), we consider the maximal t-ratio $\sup_{\omega \in \mathbb{R}} |t(\omega)|$ with

$$t(\omega) := \frac{\beta(\omega)}{\sigma(\omega)} \quad (5)$$

because it imposes all restrictions under the null, unlike the OLS and IV estimands which only impose the null for a specific choice of ω .

Assumption 1. (i) $E[\varepsilon X] = E[\varepsilon Z] = 0$, $E[X^*X] \neq 0$ and $E[Z X^*] \neq 0$

(ii) $\text{Var}(X^*) > 0$, $\text{Var}(X) > 0$, $\text{Var}(Z) > 0$, $\text{Var}(\varepsilon) > 0$

(iii) $|\text{Corr}(X, Z)| \neq 1$ and $|\text{Corr}(Y, X - Z)| \neq 1$

Notice that part (i) and (ii) of this assumption allows for endogeneity of the latent regressor ($E[\varepsilon X^*] = 0$), requiring only that the measurements X and Z are exogenous. More importantly, the assumption does not restrict the covariances ($E[X^*U]$, $E[X^*V]$, $E[UV]$). Part (iii) requires that the two measurements are not linearly dependent and Y is not perfectly correlated with the difference of the two MEs.

Theorem 1. *Suppose model (1) and Assumption 1 hold. Then, $\omega^* := \arg \max_{\omega \in \mathbb{R}^4} |t(\omega)|$ is of the form $\omega^* = (a^*, 1 - a^*, a^*, 1 - a^*)$ with*

$$a^* := \frac{E[XZ]E[Z Y] - E[Z^2]E[XY]}{E[X(Z - X)]E[Z Y] + E[Z(X - Z)]E[XY]}$$

with the possibility of $|a^*| = \infty$ if the denominator is equal to zero.

Since the maximal elements (ω_3^*, ω_4^*) are equal to (ω_1^*, ω_2^*) , the estimand for the maximal weights becomes that of the OLS regression of Y onto $W(a^*, 1 - a^*)$:

$$\beta(\omega^*) = \frac{E[W(a^*, 1 - a^*)Y]}{E[W(a^*, 1 - a^*)^2]}.$$

Notice that, in general, the weight a^* may be outside $[0, 1]$, in fact it may be equal to $\pm\infty$. Interestingly the weight depends only on observable quantities based on the distribution of (Y, X, Z) , but leads to the largest possible t-ratio regardless of the unknown measurement covariance structure $(E[X^*U], E[X^*V], E[UV])$ of the measurement system. In this sense, the maximal t-ratio adapts to the unknown measurement covariance structure without the researcher having to place a priori restrictions on those covariances.

In the special case in which X contains no information about the latent regressor X^* (i.e. X is uncorrelated with X^*), X is uncorrelated with the ME in Z , and $E[X\varepsilon] = 0$, then $E[XZ] = E[XY] = 0$ and the maximal weight becomes $a^* = 0$. This weight corresponds to OLS regression of Y onto Z , ignoring the second measurement X . On the other hand, when Z contains no information about the latent regressor X^* , Z is uncorrelated with the ME in X , and $E[Z\varepsilon] = 0$, then $E[XZ] = E[Z Y] = 0$ and the maximal weight becomes $a^* = 1$. This weight corresponds to OLS regression of Y onto X , ignoring the second measurement Z . Similarly, using the inverse function theorem one can show that there exists an a such that $\beta(a, 1 - a, a, 1 - a)$ equals the IV estimand, but the expression for that a -value is quite complicated.

Since the maximal weight ω^* can attain any value on the extended real line, it is immediately clear that the maximal t-ratio $|t(\omega^*)|$ is not smaller than the absolute value of the corresponding OLS and the IV t-ratios. Therefore, there exist data-generating processes for which the maximal t-ratio is strictly larger than any of them and one might expect that imposing all restrictions of the null by testing whether the maximal t-ratio is equal to zero might translate into favorable power properties of the test. Section 4 confirms this intuition.

Since ω^* possesses a closed-form solution, the maximal t-ratio also does. For example, when $|a^*| < \infty$, then the maximal t-ratio can be written as

$$t(\omega^*) = \text{sign}(C) \sqrt{\frac{A}{B}}$$

with

$$\begin{aligned} A &:= E [(Z E[XY] - X E[Z Y])^2] \\ B &:= A - E[Y^2] ((E[XZ])^2 - E[Z^2]E[X^2]) \\ C &:= E[X(Z - X)]E[Z Y] + E[Z(X - Z)]E[XY] \end{aligned}$$

Under the null, the t-statistic is independent of the weight, so whether the weight is finite or infinite does not change its functional form. Under alternatives $\beta \neq 0$, the case $|a^*| = \infty$ is unlikely and occurs only in extreme case in which the MEs are perfectly correlated with the true regressor X^* or the regression error ε (see Lemma 1 in the Appendix).

Now suppose there are additional, correctly measured, exogenous regressors R , $E(\varepsilon R) = 0$, so that

$$Y = \beta X^* + \gamma' R + \varepsilon. \quad (6)$$

Denote by \tilde{X} , \tilde{Z} , \tilde{Y} the residuals of regressions of X , Z , Y onto R and let $\tilde{W}(a, b) := a\tilde{X} + b\tilde{Z}$. Define $\tilde{\beta}(\omega)$ like $\beta(\omega)$, replacing $W(a, b)$ by $\tilde{W}(a, b)$, and $\tilde{\sigma}^2(\omega)$ like $\sigma^2(\omega)$, replacing $Y, W(a, b), \beta(\omega)$ by $\tilde{Y}, \tilde{W}(a, b), \tilde{\beta}(\omega)$. The t-ratio is then defined as

$$\tilde{t}(\omega) := \frac{\tilde{\beta}(\omega)}{\tilde{\sigma}(\omega)}. \quad (7)$$

The OLS estimand of regressing Y onto (X, R) corresponds to $\tilde{\beta}(1, 0, 1, 0)$, the IV estimand regressing Y onto (X, R) using (Z, R) as IVs corresponds to $\tilde{\beta}(1, 0, 0, 1)$.

The maximal t-ratio and its maximizer can then be found using the result in Theorem 1:

Corollary 1. *Suppose (6) and Assumption 1 hold with (Y, X, Z) replaced by $(\tilde{Y}, \tilde{X}, \tilde{Z})$. Then, $\tilde{\omega}^* := \arg \max_{\omega \in \mathbb{R}^4} |\tilde{t}(\omega)|$ is of the form $\tilde{\omega}^* = (\tilde{a}^*, 1 - \tilde{a}^*, \tilde{a}^*, 1 - \tilde{a}^*)$ with*

$$\tilde{a}^* := \frac{E[\tilde{X}\tilde{Z}]E[\tilde{Z}\tilde{Y}] - E[\tilde{Z}^2]E[\tilde{X}\tilde{Y}]}{E[\tilde{X}(\tilde{Z} - \tilde{X})]E[\tilde{Z}\tilde{Y}] + E[\tilde{Z}(\tilde{X} - \tilde{Z})]E[\tilde{X}\tilde{Y}]}$$

with the possibility of $|\tilde{a}^*| = \infty$ if the denominator is equal to zero.

The t-ratios defined in (5) and (7) use asymptotic variance expressions for homoskedastic regression errors. We now consider conditional heteroskedasticity in the regression model (6), i.e. when the conditional variance of the regression error given X^* is not independent of X^* . In this case, we are interested in the t-ratio

$$\bar{t}(\omega) := \frac{\tilde{\beta}(\omega)}{\bar{\sigma}(\omega)},$$

where the variance is

$$\bar{\sigma}^2(\omega) := \frac{E[\tilde{\varepsilon}(\omega)^2 \tilde{W}(\omega_1, \omega_2)^2]}{(E[\tilde{W}(\omega_1, \omega_2) \tilde{W}(\omega_3, \omega_4)])^2}$$

with $\tilde{\varepsilon}(\omega) := \tilde{Y} - \tilde{\beta}(\omega) \tilde{W}(\omega_3, \omega_4)$.

Without any further restrictions on the form of heteroskedasticity, there does not in general exist a closed-form expression for the maximal weights nor the maximal t-ratio. However, we can show that the four-dimensional optimization problem over ω can be reduced to a two-dimensional one, which simplifies implementation significantly.

Theorem 2. *Suppose model (6) and Assumption 1 hold with (Y, X, Z) replaced by $(\tilde{Y}, \tilde{X}, \tilde{Z})$. Then, for any $\omega := (\omega_1, \omega_2, \omega_3, \omega_4) \in \mathbb{R}^4$, there exist $a_1, a_2 \in \mathbb{R}$ such that, for $\omega' := (a_1, 1 - a_1, a_2, 1 - a_2)$,*

$$\bar{t}(\omega) = \bar{t}(\omega').$$

Remark 1. *In principle, one could construct a test statistic for H_0 using more complicated, non-linear combinations of X and Z rather than the linear combinations as in the definition of $W(a, b)$. For example, one could consider $W(a, b) := a'p(X) + b'q(Z)$, where $a, b \in \mathbb{R}^K$, and $p(\cdot)$ and $q(\cdot)$ are K -dimensional vectors of basis functions (e.g. polynomials or splines). The results of this section can easily be extended to this case. However, allowing for interaction terms between the basis functions of X and Z would require more involved arguments and we leave these for future work. In this paper, we focus on linear combinations because we place more importance on practical usefulness of the method and because we want to avoid the maximal t-statistic to depend on higher-order moments of (Y, X, Z) which would make it more sensitive to outliers. \square*

2.2 Testing General Hypotheses

In this section, we consider testing the more general hypothesis

$$H_0 : \beta = b \quad \text{vs.} \quad H_1 : \beta \neq b \tag{8}$$

for some $b \in \mathbb{R}$ in the simple regression model (1). Under the null of no effect, (2), the estimand $\beta(a)$ considered in the previous section is zero for all weights a . This is not the case under the more general null, i.e. under (8) the estimand $\beta(a)$ is not equal to b for all weights a because β is not identified. However, one can construct bounds for β based on observable quantities. To describe these bounds consider the following notation. Because of Theorem 2 it suffices to consider linear combinations of measurements with weights a and $1 - a$, so we simplify the notation to $W(a) := aX + (1 - a)Z$. Let

$$\beta_{OLS}(a) := \frac{E[W(a)Y]}{E[W(a)^2]}$$

be the OLS estimand of a regression of Y on $W(a)$, and

$$\beta_{OLS-INV}(a) := \frac{E[Y^2]}{E[W(a)Y]}$$

that of the reverse regression. Similarly, the IV estimand from a regression of Y on $W(a)$ using $W(b)$ as an instrument is

$$\beta_{IV}(a, b) := \frac{E[W(b)Y]}{E[W(b)W(a)]}.$$

Assumption 2. $E[\varepsilon X^*] = E[\varepsilon U] = E[\varepsilon V] = 0$

Assumption 3. $E[X^*X] > 0$ and $E[X^*Z] > 0$

Assumption 4. $E[X^*X] \geq E[XZ] > 0$ and $E[X^*Z] \geq E[XZ] > 0$

Assumption 5. $E[UX] \geq 0$ and $E[VZ] \geq 0$

Assumption 6. $E[X^*U] \leq 0$ and $E[X^*V] \leq 0$

For the discussion in this section, suppose $\beta \geq 0$ (similar results hold for the case $\beta \leq 0$). Black, Berger, and Scott (2000) show how to construct bounds on β using the above assumptions.¹ First, the OLS estimand and the inverse OLS estimand form lower and upper bounds on the regression coefficient:

$$\max_{a=0,1} \beta_{OLS}(a) \leq \beta \leq \min_{a=0,1} \beta_{OLS-INV}(a).$$

¹Lemma 2 in the appendix provides a slightly modified statement and derivation for those bounds.

The IV estimand provides an additional bound that may or may not be tighter than the OLS bound,

$$\beta \leq \min_{a=0,1} \beta_{IV}(a, 1 - a).$$

We now show that the bounds cannot be refined by considering linear combinations of measurements.

Theorem 3. *Suppose $\beta \geq 0$. Then:*

1. *Under Assumptions 2, 3, and 5, $\beta_{OLS}(a) \leq \beta$ holds only for $a \in \{0, 1\}$.*
2. *Under Assumptions 2, 3, and 6, $\beta \leq \beta_{OLS-INV}(a)$ holds only for $a \in [0, 1]$.*
3. *Under Assumptions 2 and 4, $\beta \leq \beta_{IV}(a, 1 - a)$ holds only for $a \in \{0, 1\}$.*

The first and third statement of this theorem imply that it is not possible to refine the OLS and IV bounds using linear combinations of measurements. The proof shows that, for any $a \notin \{0, 1\}$, we can find a data-generating process so that the bounds are violated. The second part of the theorem states that the inverse OLS bound holds for all $a \in [0, 1]$. However, because the inverse OLS estimand is inversely related to a linear function of the weight a , it is easy to see that the tightest bound, $\min_{a \in [0,1]} \beta_{OLS-INV}(a)$, is achieved by either $a = 0$ or $a = 1$. Therefore, the inverse OLS bound also cannot be refined by considering linear combinations of the two measurements.

Under the null hypothesis of no effect, (2), the estimand $\beta(a)$ is equal to zero for all weights a , but under the alternative it varies with a . This is why it was possible to increase the t-statistic by maximizing over a . In the case of testing the general null (8), however, the estimand of interest is the set of possible β values consistent with the data,

$$\Theta(a_1, a_2, a_3) := \{\beta \in \mathbb{R}: \beta_{OLS}(a_1) \leq \beta \leq \min\{\beta_{OLS-INV}(a_2), \beta_{IV}(a_3, 1 - a_3)\}\}.$$

These sets contain the identified set for β and thus are valid outer sets for any $(a_1, a_2, a_3) \in \{0, 1\}^3$. Testing (8) requires checking whether b is in the intersection of these sets. Since the sets are not valid outer sets of the identified set for any value of $(a_1, a_2, a_3) \notin \{0, 1\}^3$, we cannot take linear combinations of measurements to find a maximal statistic. Therefore, the approach of the previous section does not extend to the general hypothesis.

3 The Maximal t-Test

In this section, we propose a powerful t-test of the hypothesis (2), making use of the maximal combination of two measurements. We want to allow for heteroskedasticity-robust variance estimators in the construction of the t-statistic. As we have shown in Theorem 2, for the calculation of the maximal t-statistic, we do not have to consider general weights ω in \mathbb{R}^4 , but only weights of the form $(a_1, 1 - a_1, a_2, 1 - a_2)$. A direct implementation of this result could consider the test statistic

$$\max_{(a_1, a_2) \in \mathbb{R}^2} |\hat{t}(a_1, 1 - a_1, a_2, 1 - a_2)|,$$

where \hat{t} is an estimator of t multiplied by \sqrt{n} . It is easy to show that, under standard conditions, $\hat{t}(\cdot)$ weakly converges to a Gaussian process, so that by the delta method the maximal t-statistic converges to the supremum of that limiting process. Unfortunately, it is difficult to construct critical values from this process because its covariance function depends upon the unknown data-generating process. We therefore propose a simple multiplier bootstrap method to calculate critical values.

Suppose we observe an i.i.d. sample $\mathcal{Y}_n := \{(Y_i, X_i, Z_i)\}_{i=1}^n$ from the distribution of (Y, X, Z) . We describe a procedure that applies to numerical optimization of the t-ratio. Define a grid of weights $\omega_1, \dots, \omega_p$ of the form $\omega_j = (a_{j,1}, 1 - a_{j,1}, a_{j,2}, 1 - a_{j,2})'$ for some $(a_{j,1}, a_{j,2})' \in \mathbb{R}^2$. Let $W_{1ij} := \omega_{j,1}X_i + \omega_{j,2}Z_i$, $W_{2ij} := \omega_{j,3}X_i + \omega_{j,4}Z_i$, $\sigma_j^2 := E[\varepsilon_i^2 W_{1ij}^2]$, and $\hat{\sigma}_j^2$ some estimator of σ_j^2 . The notation $\mathbb{E}_n[\cdot]$ denotes the average over the index $1 \leq i \leq n$, so for example $\mathbb{E}_n[x_{ij}] = n^{-1} \sum_{i=1}^n x_{ij}$.

Our maximal t-statistic is defined as $|\hat{t}(\omega)|$ optimized over the grid $\omega_1, \dots, \omega_p$:

$$T := \max_{\omega \in \{\omega_1, \dots, \omega_p\}} |\hat{t}(\omega)| = \max_{1 \leq j \leq p} \left| \frac{\sqrt{n} \mathbb{E}_n[W_{1ij} Y_i]}{\hat{\sigma}_j} \right|.$$

We now describe the construction of critical values using a multiplier bootstrap. Let $\{e_i^b\}_{i=1}^n$, $b = 1, \dots, B$, be an i.i.d. sequence of standard normally distributed random variables that are independent of \mathcal{Y}_n , and define the residual $\hat{\varepsilon}_{ij} := Y_i - \hat{\beta}_j W_{2ij}$ where $\hat{\beta}_j$ is the IV estimator from a regression of Y_i onto W_{2ij} using W_{1ij} as instrument. We define the bootstrap statistic

$$T^b := \max_{1 \leq j \leq p} \left| \frac{\sqrt{n} \mathbb{E}_n[e_i W_{1ij} \hat{\varepsilon}_{ij}]}{\hat{\sigma}_j} \right|, \quad b = 1, \dots, B,$$

and the bootstrap critical value

$$c_\alpha := \text{conditional } (1 - \alpha) - \text{quantile of } T^b \text{ given the data } \mathcal{Y}_n.$$

We reject H_0 at the nominal level α if and only if $T > c_\alpha$.

The validity of this bootstrap critical value can be shown using the high-dimensional central limit theorem in Corollary 3.1 of [Chernozhukov, Chetverikov, and Kato \(2013\)](#). It implies that the test has limiting rejection probability under the null equal to the nominal level α and is consistent against any fixed alternative violating the null. The result requires only mild conditions on the data-generating process and, in particular, allows the vector W_{1ij} to grow at a rate that is an exponential function of the sample size. This means that we can consider grids for the weights $\omega_1, \dots, \omega_p$ of very large size p .

4 Simulations

This section studies the finite sample performance of the maximal t-test described in Section 2.2. We consider the simple regression model in (1) with the two measurements X and Z generated from $X = X^* + U$ and $Z = X^* + V$ with

$$\begin{bmatrix} X^* \\ U \\ V \end{bmatrix} \sim N \left(\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & \sigma_{X^*U} & \sigma_{X^*V} \\ \sigma_{X^*U} & \sigma_U^2 & \sigma_{UV} \\ \sigma_{X^*V} & \sigma_{UV} & \sigma_V^2 \end{bmatrix} \right)$$

and $\varepsilon \sim N(0, 1)$ is independent of (X^*, U, V) . The null hypothesis holds with $\beta = 0$. To generate alternatives we increase β on a grid up to 0.8. We define six different scenarios in which we vary the parameters σ_U^2 , σ_V^2 , σ_{UV} . Table 1 provides the definition of those scenarios. Within each of these we vary the covariances σ_{X^*U} and σ_{X^*V} . We generate 1,000 Monte Carlo samples of size $n = 200$.

We consider four tests. The first two are t-tests based on the OLS estimator from a regression of Y on X (“OLS w/ x ”) and of Y on Z (“OLS w/ z ”), respectively. The third (“IV $x|z$ ”) is the t-test based on the IV estimator from a regression of Y on X using Z as instrument. The fourth (“tmax”) is the maximal t-test with the multiplier bootstrap critical value described in Section 2.2. We use $B = 1,000$ bootstrap samples and an equally spaced grid of weights in the interval $[0, 1]$ with a distance of 0.2 between

Scenario	σ_U^2, σ_V^2	σ_{UV}	σ_{X*U}	σ_{X*V}
1	2	0	-0.3 or -0.7	-0.3 or -0.7
2	2	0.5	-0.3 or -0.7	-0.3 or -0.7
3	2	-0.5	-0.3 or -0.7	-0.3 or -0.7
4	1	0	-0.3 or -0.5	-0.3 or -0.5
5	1	0.3	-0.3 or -0.5	-0.3 or -0.5
6	1	-0.3	-0.3 or -0.5	-0.3 or -0.5

Table 1: Parameter values in each scenario.

the grid points. All tests are implemented with the homoskedastic variance estimator and nominal level of 0.05.

Table 2 provides the null rejection frequencies of the tests and Figures 1–6 the power curves for each of the six scenarios.

All tests control size well as expected. The power, however, varies significantly among them. The findings can be summarized as follows. First, in almost all scenarios and parameter combinations, the IV-based t-test performs poorest. Second, in none of the scenarios and parameter combinations, the maximal t-test is dominated by another test. In most of them, the maximal t-test strictly dominates all other tests (e.g. in scenarios 3 and 6). When the maximal t-test does not strictly dominate all other tests, then it does strictly dominate all but one of the tests, with the latter possessing power roughly equal to that of the maximal t-test. In most of the scenarios and parameter combinations, the power gains of the maximal t-test relative to the others is significant (e.g. scenarios 3 and 6). Third, a common feature among all simulation results is that when the measurement X is more precise than Z , then the OLS-based test using X dominates the OLS-based test using Z and the former is closer to the maximal t-test in terms of power (compare, for example, the upper-left and the lower-right panels in Figure 1). Our maximal t-test either dominates both or is equal to the more powerful of the two tests. In this sense, it automatically adapts to the unknown covariance structure in the measurement system and puts more weight on the more informative measurement.

σ_{X*U}	σ_{X*V}	Test	S1	S2	S3	S4	S5	S6
strong	strong	OLS w/ x	0.054	0.050	0.055	0.040	0.046	0.048
		OLS w/ z	0.039	0.038	0.052	0.050	0.059	0.039
		IV x z	0.018	0.000	0.051	0.000	0.035	0.023
		tmax	0.051	0.050	0.069	0.042	0.056	0.058
		a^*	0.507	0.520	0.482	0.491	0.480	0.483
strong	weak	OLS w/ x	0.049	0.051	0.043	0.056	0.049	0.058
		OLS w/ z	0.050	0.062	0.058	0.052	0.033	0.046
		IV x z	0.001	0.027	0.024	0.010	0.028	0.001
		tmax	0.059	0.056	0.056	0.056	0.047	0.057
		a^*	0.529	0.509	0.512	0.508	0.507	0.530
weak	strong	OLS w/ x	0.037	0.046	0.042	0.058	0.049	0.042
		OLS w/ z	0.051	0.049	0.047	0.055	0.062	0.052
		IV x z	0.001	0.025	0.016	0.006	0.052	0.001
		tmax	0.048	0.057	0.055	0.061	0.064	0.057
		a^*	0.470	0.476	0.477	0.509	0.465	0.461
weak	weak	OLS w/ x	0.051	0.059	0.050	0.048	0.049	0.040
		OLS w/ z	0.058	0.044	0.052	0.058	0.049	0.049
		IV x z	0.015	0.034	0.000	0.035	0.046	0.002
		tmax	0.067	0.056	0.059	0.063	0.051	0.054
		a^*	0.471	0.530	0.511	0.475	0.515	0.502

Table 2: Rejection probabilities in the six different scenarios. The values of σ_{X*U} and σ_{X*V} vary between “strong” (-0.7 for scenarios 1–3; -0.5 for scenarios 4–6) and “weak” (-0.3 for all scenarios). a^* denotes the maximal weight averaged across the MC samples.

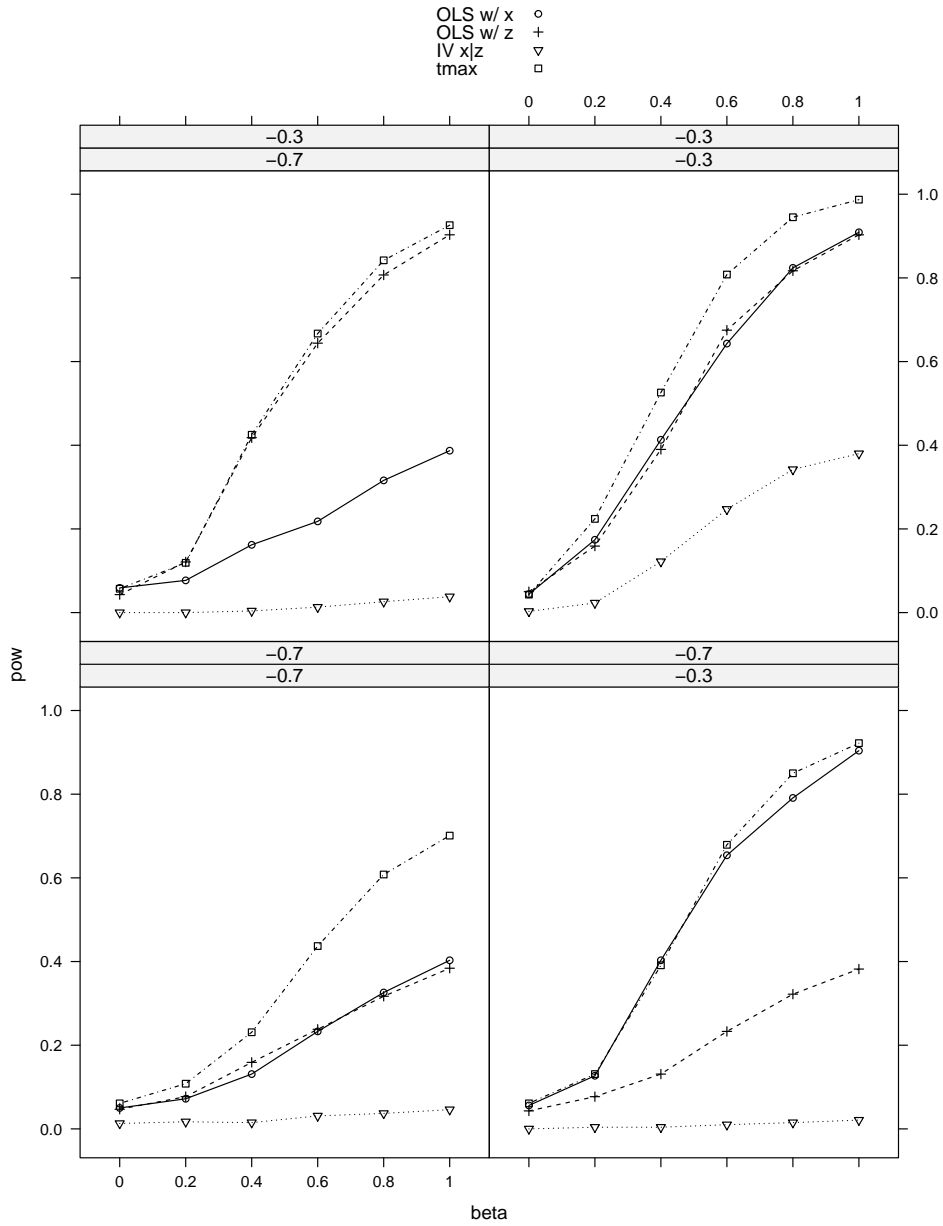


Figure 1: Rejection probabilities in scenario 1, for $\sigma_{X*V} \in \{-0.3, -0.7\}$ varying from top to bottom and $\sigma_{X*U} \in \{-0.3, -0.7\}$ varying from left to right.

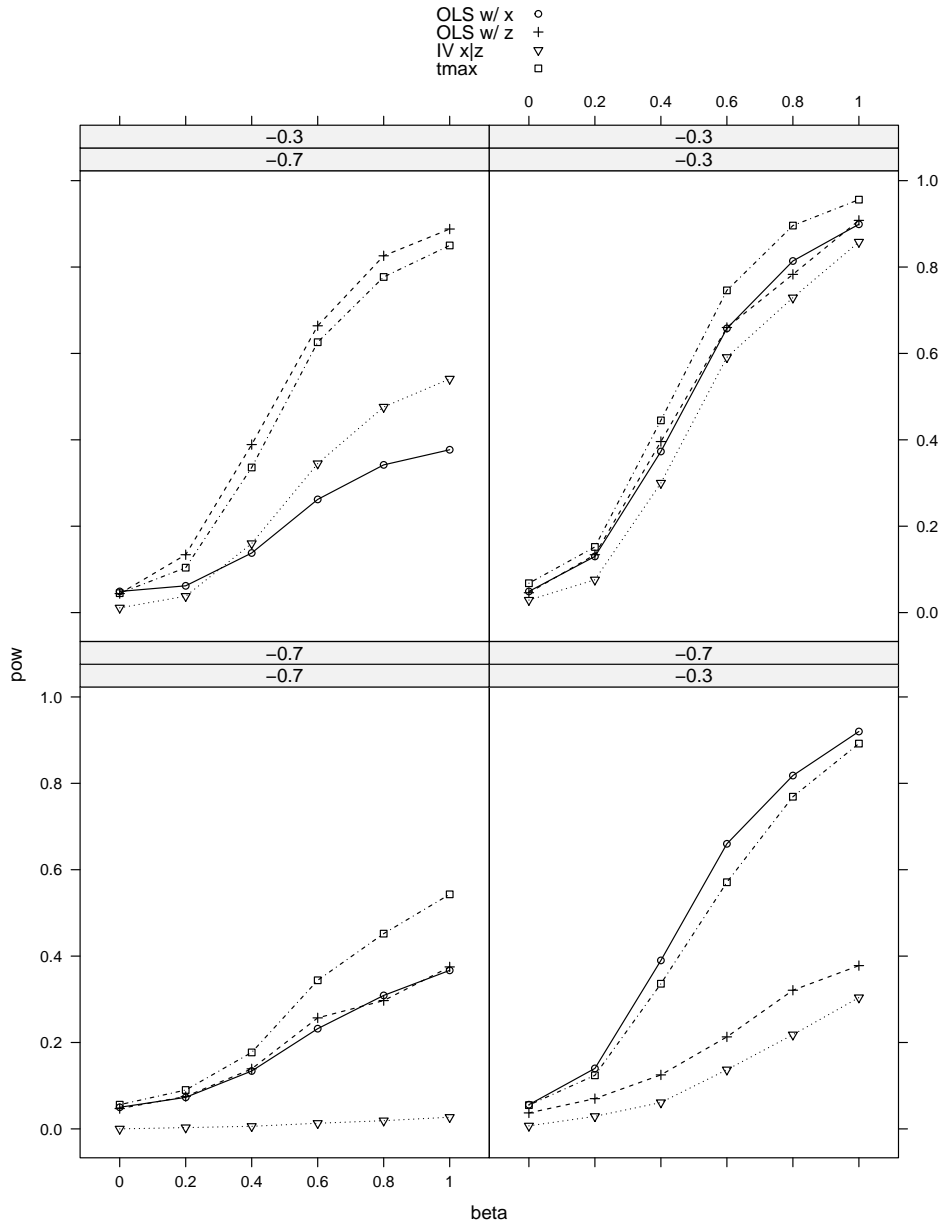


Figure 2: Rejection probabilities in scenario 2, for $\sigma_{X*V} \in \{-0.3, -0.7\}$ varying from top to bottom and $\sigma_{X*U} \in \{-0.3, -0.7\}$ varying from left to right.

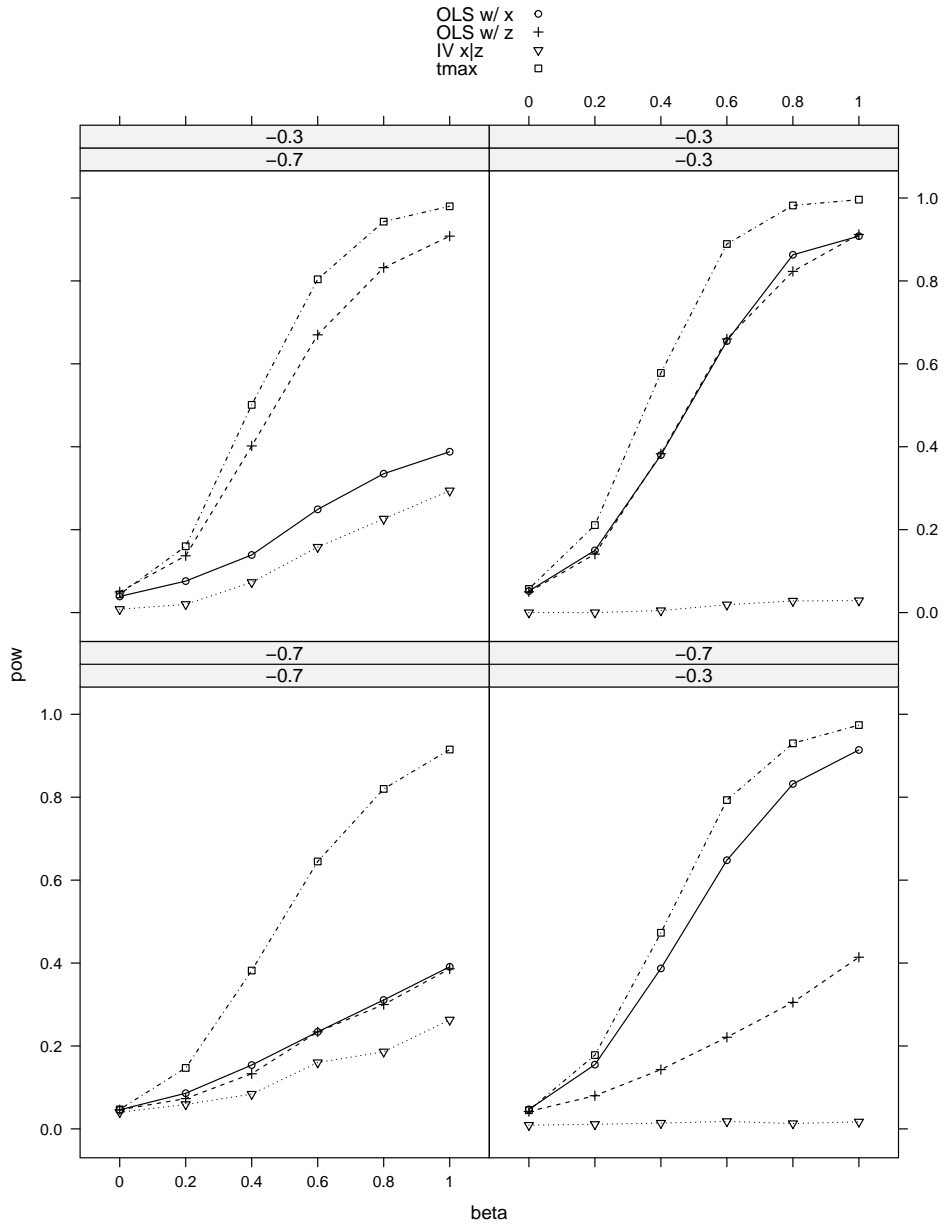


Figure 3: Rejection probabilities in scenario 3, for $\sigma_{X*V} \in \{-0.3, -0.7\}$ varying from top to bottom and $\sigma_{X*U} \in \{-0.3, -0.7\}$ varying from left to right.

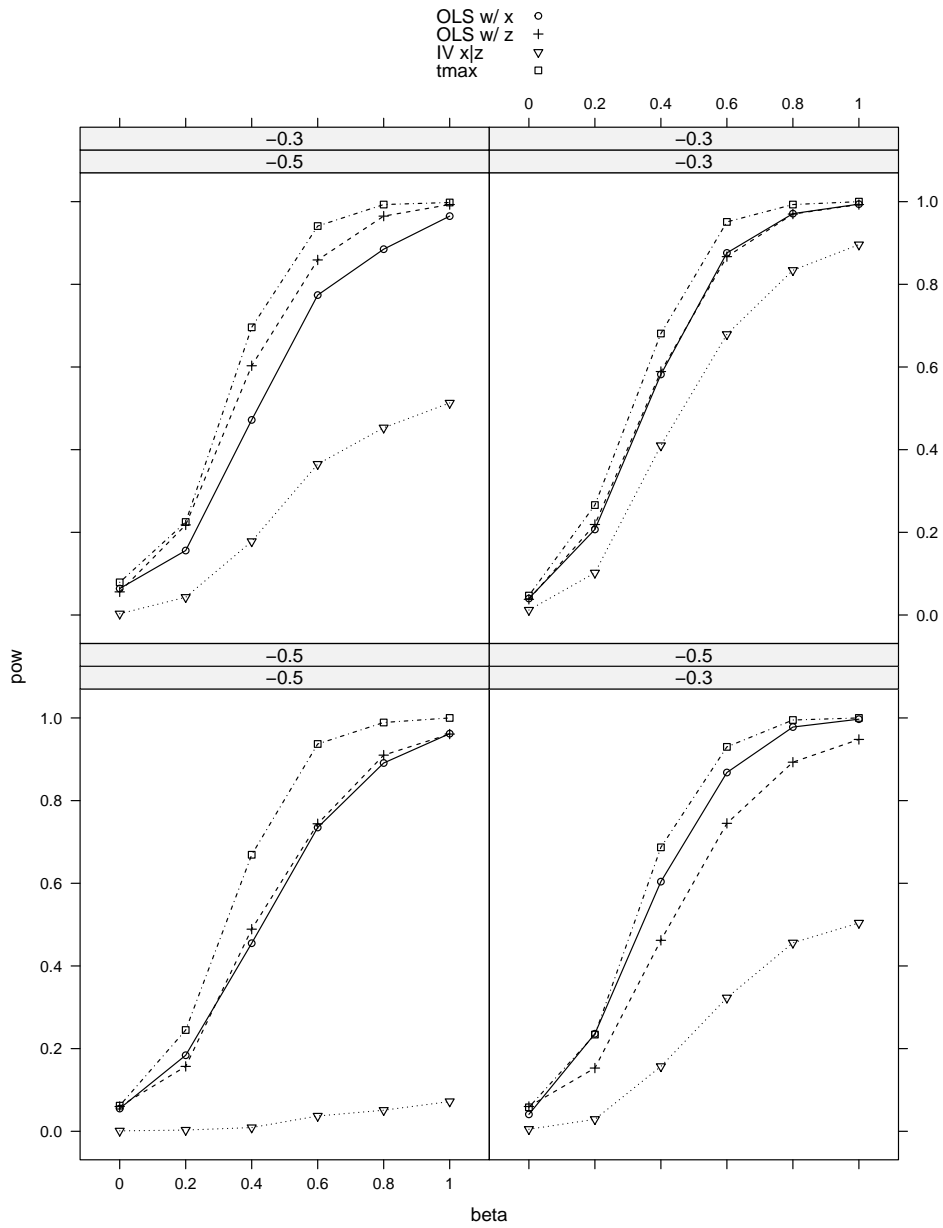


Figure 4: Rejection probabilities in scenario 4, for $\sigma_{X*V} \in \{-0.3, -0.5\}$ varying from top to bottom and $\sigma_{X*U} \in \{-0.3, -0.5\}$ varying from left to right.

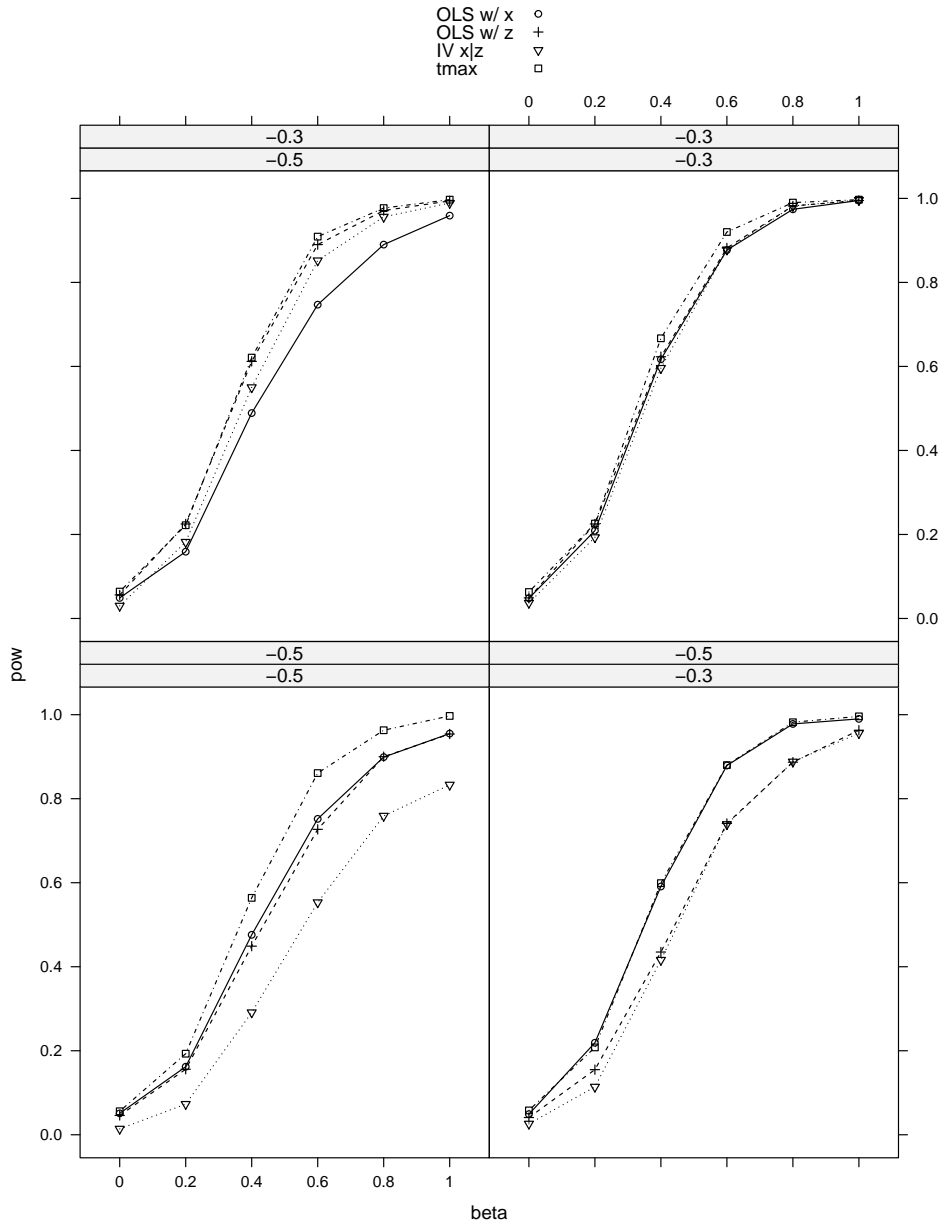


Figure 5: Rejection probabilities in scenario 5, for $\sigma_{X*V} \in \{-0.3, -0.5\}$ varying from top to bottom and $\sigma_{X*U} \in \{-0.3, -0.5\}$ varying from left to right.

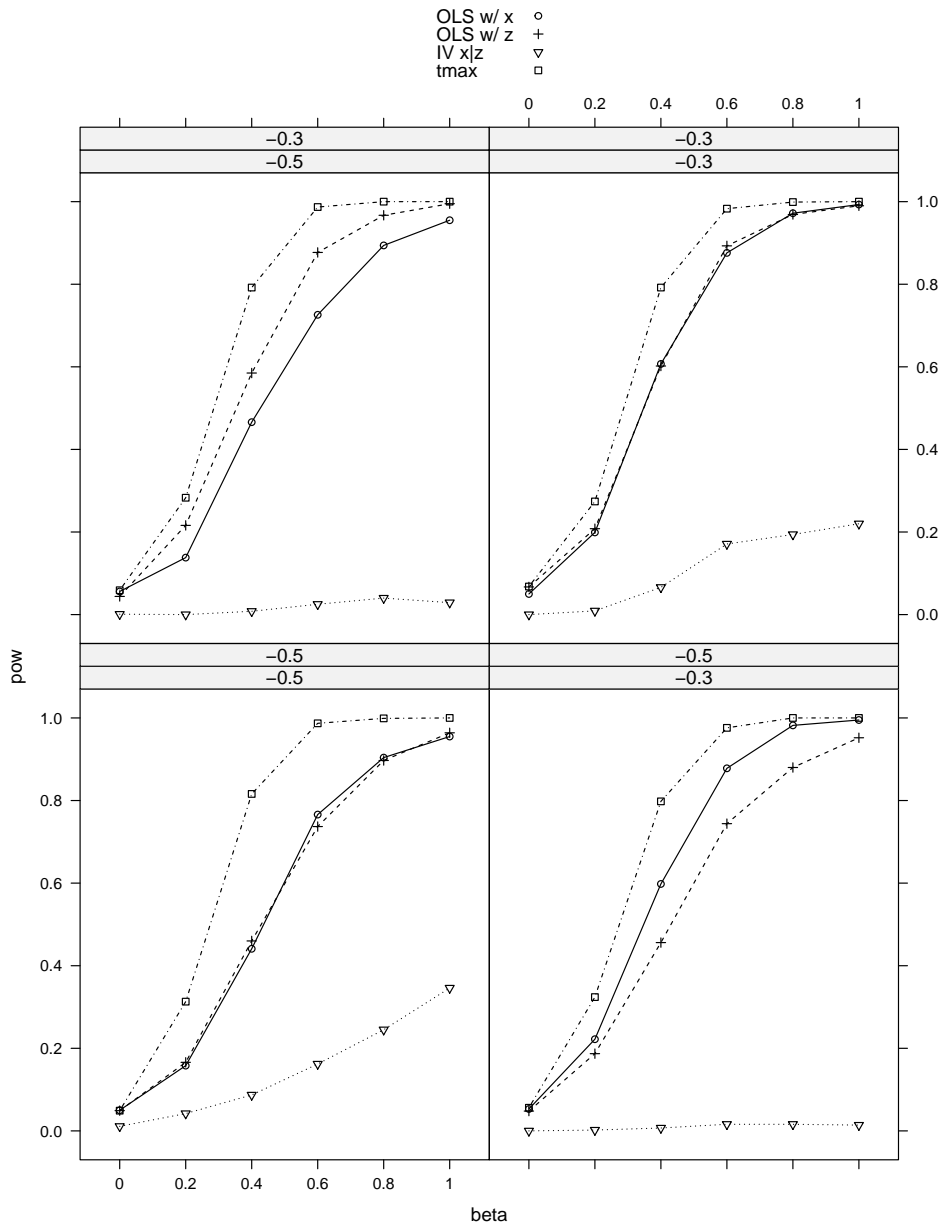


Figure 6: Rejection probabilities in scenario 6, for $\sigma_{X*V} \in \{-0.3, -0.5\}$ varying from top to bottom and $\sigma_{X*U} \in \{-0.3, -0.5\}$ varying from left to right.

5 Empirical Application

In this section, we revisit the empirical work by [Ashenfelter and Krueger \(1994\)](#) who studied the returns to schooling using data from monozygotic twins surveyed at the 16th Annual Twins Days Festival. For a pair of twins, let Y_1 and Y_2 be log wages of the first and second twin. Denote by X_1 and X_2 their respective self-reported levels of education. In addition, the second twin is asked to report the education of the first twin, Z_1 , and the first to report the education of the second, Z_2 . We consider the regression model (1) with $Y = Y_1 - Y_2$ the difference of log wages, X^* the difference of true education levels, $X = X_1 - X_2$ the difference of self-reported education, and $Z = Z_1 - Z_2$ the difference of cross-reported education.

First, we test the null of no ME in self-reported education levels using the nonparametric test in [Wilhelm \(2017\)](#).² The test statistic is 0.00060 and the critical value 0.00156, leading to a p-value of 0.376. In conclusion, there is not enough information in the dataset to reject the null of no ME in self-reported education. This could simply be because the sample size (147) is too small so the nonparametric test does not have enough power. Therefore, in a second step we impose the linearity of the wage equation (which was proposed in [Ashenfelter and Krueger \(1994\)](#)) and perform the test of no ME again. To do this we regress Y onto both X and Z (and a constant) and find that the coefficient of Z is statistically significant at the 5% level (t-statistic of 2.21). Therefore, with the assumption of linearity, we reject the null of no ME in self-reported education ([Hausman \(1978\)](#) and [Wilhelm \(2017\)](#)). This finding is consistent with the possibility that the nonparametric test has low power because of the small sample size and thus fails to reject, but the parametric test that imposes linearity of the conditional expectation is more powerful and thus does reject.

Since the education variable is discrete, we know that by construction ME (if any) cannot be classical because it has to depend on the true level of education. Furthermore, the MEs in X and Z are likely correlated because the twins may have a similar “ability to report correctly” due to identical gene pool, family and social background. These features of the data-generating process make it impossible to identify and consistently estimate the returns to education unless strong assumptions (e.g. parametrization of all unobservable

²We implement the test based on [Delgado and Manteiga \(2001\)](#)’s Cramér-von Mises statistic for the null hypothesis (15) in [Wilhelm \(2017\)](#) with $\mu(y) = y$, a gaussian kernel, and 1,000 bootstrap replications.

distributions) are imposed. However, testing the null of no effect is possible under the weak regularity conditions in Assumption 1.

We now apply our maximal t-test to the data set. To evaluate the power properties, we use subsamples of the data set. The sample size in each trial varies from 30 to 147 (the full sample). Table 3 shows the test results for four tests: the t-test based on the OLS estimator from a regression of Y on X (“OLS w/ x ”), the t-test based on the OLS estimator from a regression of Y on Z (“OLS w/ z ”), the t-test based on the IV estimator from a regression of Y on X using Z as instrument (“IV”), and the maximal t-test proposed in Section 2.2 (“tmax”). The table also shows the maximizer of the t-statistic (“ \hat{a}^* ”) and the multiplier bootstrap critical value (“CV”) for a nominal level of 0.05. We implement our maximal t-test with an equally spaced grid of weights in the interval $[0, 1]$ with a distance of 0.1 and $B = 1,000$ bootstrap replications.³

n	OLS w/ x	OLS w/ z	IV	\hat{a}^*	tmax	CV
30	1.39	3.03	1.36	0.2	3.22	2.85
40	1.61	2.99	1.90	0.2	3.15	2.74
50	1.61	3.20	1.94	0.2	3.34	2.75
60	1.82	3.91	2.44	0.2	4.00	2.81
70	1.60	3.22	2.64	0.1	3.25	2.62
80	1.75	3.18	2.72	0.1	3.22	2.41
90	1.99	3.17	2.76	0.2	3.26	2.49
100	2.21	3.34	2.98	0.2	3.43	2.61
110	2.49	3.40	3.09	0.3	3.57	2.70
120	3.09	3.61	3.38	0.4	3.93	2.45
130	3.25	3.54	3.36	0.5	3.95	2.59
147	3.69	3.86	3.72	0.5	4.34	2.42

Table 3: Tests applied to first n observations of the sample; rejections at the 5% level are highlighted in bold face.

Our maximal t-test rejects the null of no effect for all sample sizes, even when $n = 30$. Ashenfelter and Krueger (1994) consider the OLS estimator using X and the IV estimator. These do reject for the full sample, but not for all smaller samples. The OLS-based test

³The results remain the same when the grid is defined in $[-2, 2]$ with a distance of 0.01.

rejects for sample sizes of 90 or larger and the IV-based test for sizes of 60 or larger. This finding is consistent with our simulations in which the maximal t-test tends to be more powerful than any of the other tests. The OLS estimator using Z was not considered in [Ashenfelter and Krueger \(1994\)](#), but it turns out to lead to a t-test that also rejects for all sample sizes.

The findings in this empirical application are consistent with the simulation results. They could, for example, be interpreted similarly to the upper-left panel of [Figure 2](#). The maximal t-test rejects the null at all sample sizes and so does OLS w/z. The other two tests fail to reject the null at smaller sample sizes but the IV test shows stronger power. It is plausible that twins may have similar “ability to report correctly” in the sense that U and V are positively correlated. As the OLS-based test using Z seems to reject more often than the one using X , the findings are consistent with Z being a more precise measurement than X . The covariance structure of (X^*, U, V) is not identified and thus cannot be consistently estimated, so knowing a priori which measurement is more precise, whether linear combinations of measurements help improve power, and therefore which of the many different possible OLS and IV tests to use cannot be known. However, our maximal t-test adapts to the unknown covariance structure, thus does not require such knowledge, and is more powerful.

6 Conclusion

There are some interesting directions for future research. For example, it would be useful to formally derive the power of our maximal t-test and compare it to the standard OLS- and IV-based t-tests. This is a nonstandard problem because our test statistic is not asymptotically normal. More importantly, the maximal t-statistic is weakly larger than the standard OLS- and IV-based t-statistics (for most data-generating processes it is strictly larger), but the corresponding critical value is also weakly larger. It is therefore not straightforward to formally compare power. Another direction would be to extend our maximal t-test to nonlinear models (e.g. moment condition models), which is important for the estimation of production function with latent inputs, for example.

A Finiteness of the Optimal Weights

Assumption 7. (i) $E[XY] \neq 0$, $E[ZY] \neq 0$, and (ii) $|\text{corr}(X^*, (Z - X))| \neq 1$.

The possibility of $|a^*| = \infty$ can be ruled out under a set of weak assumptions. The first part implies that $\beta \neq 0$. It also rules out the cases in which the MEs are perfectly negatively correlated with the true variable ($E[(X^*)^2] = -E[X^*U] = -E[X^*V]$). The last part means that the true variable is not recovered from $Z - X$.

Lemma 1. *Suppose (1) and Assumption 1 and 7 hold. Then, the maximal weight a^* is finite.*

$$|a^*| < \infty$$

Proof of Lemma 1. The numerator of a^* in Theorem 1 is finite if the denominator is nonzero, then $|a^*| < \infty$.

$$C := E[X(Z - X)]E[Z Y] - E[Z(Z - X)]E[XY]$$

By Assumption 1 (i), $E[XY] = \beta E[XX^*]$ and $E[Z Y] = \beta E[Z X^*]$. Then C trivially becomes zero when $\beta = 0$ or $E[(X^*)^2] + E[X^*U] = E[(X^*)^2] + E[X^*V] = 0$. These are ruled out by Assumption 7. The other cases in which C is zero are following.

1. $E[Z(Z - X)] = 0$ and $E[X(Z - X)] = 0$
2. $\frac{E[X(Z - X)]}{E[Z(Z - X)]} = \frac{E[XY]}{E[Z Y]} = \frac{E[(X^*)^2] + E[X^*U]}{E[(X^*)^2] + E[X^*V]}$

The first case implies that $X = Z$ and is ruled out by (iii) of Assumption 1. The second case indicates that the ratio of $E[X(Z - X)]$ to $E[Z(Z - X)]$ is equal to the ratio of $E[(X^*)^2] + E[X^*U]$ to $E[(X^*)^2] + E[X^*V]$. Therefore, for a constant $\gamma \neq 0$,

$$E[XZ - X^2] = \gamma(E[(X^*)^2] + E[X^*U]), \quad E[Z^2 - XZ] = \gamma(E[(X^*)^2] + E[X^*V]).$$

By simple algebra,

$$E[X^*V] - (1 + \gamma)E[X^*U] + E[UV] - E[U^2] = \gamma E[(X^*)^2]$$

$$(1 - \gamma)E[X^*V] - E[X^*U] + E[V^2] - E[UV] = \gamma E[(X^*)^2]$$

and therefore,

$$\gamma(E[X^*V] - E[X^*U]) = E[V^2] - 2E[UV] + E[U^2]$$

and this implies that

$$\gamma E[X^*(V - U)] = E[(V - U)^2] \implies X^* = \frac{1}{\gamma}(V - U) = \frac{1}{\gamma}(Z - X).$$

As Assumption 7 rules out perfect positive and negative correlation between X^* and $Z - X$, such a possibility is ruled out. Therefore, the denominator C is non-zero and hence a^* is finite. Q.E.D.

B Proofs

B.1 Proofs for Section 1

Proof of Theorem 1. First, we show that

$$\sup_{\omega \in \mathbb{R}^4} t(\omega)^2 = \sup_{a \in \mathbb{R}} t(a, 1 - a, a, 1 - a)^2. \quad (9)$$

To this end, notice that when $(\omega_3, \omega_4) = (\omega_1, \omega_2)$

$$\beta(\omega_1, \omega_2, \omega_1, \omega_2) = \frac{E[W(\omega_1, \omega_2)Y]}{E[W(\omega_1, \omega_2)^2]}, \quad \sigma^2(\omega_1, \omega_2, \omega_1, \omega_2) = \frac{\sigma_\varepsilon^2(\omega_1, \omega_2, \omega_1, \omega_2)}{E[W(\omega_1, \omega_2)^2]}$$

and the t-ratio becomes

$$t(\omega_1, \omega_2, \omega_1, \omega_2) = \frac{E[W(\omega_1, \omega_2)Y]}{\sigma_\varepsilon(\omega_1, \omega_2, \omega_1, \omega_2) \sqrt{E[W(\omega_1, \omega_2)^2]}}.$$

Since $\beta(0, 0, \omega_3, \omega_4) = 0$ for any $\omega_3, \omega_4 \in \mathbb{R}^2$, $\omega_1 = \omega_2 = 0$ cannot be a maximizer of the t-ratio. Therefore, in the remainder of the proof we assume without loss of generality that $(\omega_1, \omega_2) \neq (0, 0)$. By Assumption 1(ii), there is no linear combination of X and Z that is equal to zero so that, together with the positive variance Assumption 1(i), we have $E[W(a, b)^2] \neq 0$ for all $(a, b) \in \mathbb{R}^2 \setminus \{(0, 0)\}$. Therefore, the residual variance $E[(Y - \beta W(\omega_1, \omega_2))^2]$ is minimized at the OLS estimand $\beta = E[W(\omega_1, \omega_2)Y]/E[W(\omega_1, \omega_2)^2]$, which in turn implies $t^2(\omega_1, \omega_2, \omega_3, \omega_4) \leq t^2(\omega_1, \omega_2, \omega_1, \omega_2)$ for all $\omega \in \mathbb{R}^4$, $(\omega_1, \omega_2) \neq (0, 0)$, with equality when $(\omega_1, \omega_2) = (\omega_3, \omega_4)$. Therefore,

$$\sup_{\omega \in \mathbb{R}^4} t^2(\omega_1, \omega_2, \omega_3, \omega_4) = \sup_{(\omega_1, \omega_2) \in \mathbb{R}^2} t^2(\omega_1, \omega_2, \omega_1, \omega_2) \quad \forall \omega \in \mathbb{R}^4.$$

Next, notice that

$$\beta \left(\frac{\omega_1}{\omega_1 + \omega_2}, \frac{\omega_2}{\omega_1 + \omega_2}, \frac{\omega_1}{\omega_1 + \omega_2}, \frac{\omega_2}{\omega_1 + \omega_2} \right) = \beta(\omega_1, \omega_2, \omega_1, \omega_2)(\omega_1 + \omega_2)$$

and

$$\sigma^2 \left(\frac{\omega_1}{\omega_1 + \omega_2}, \frac{\omega_2}{\omega_1 + \omega_2}, \frac{\omega_1}{\omega_1 + \omega_2}, \frac{\omega_2}{\omega_1 + \omega_2} \right) = \sigma^2(\omega_1, \omega_2, \omega_1, \omega_2)(\omega_1 + \omega_2)^2.$$

Therefore, the weights in the t-ratio can be standardized to sum up to one,

$$t^2 \left(\frac{\omega_1}{\omega_1 + \omega_2}, \frac{\omega_2}{\omega_1 + \omega_2}, \frac{\omega_1}{\omega_1 + \omega_2}, \frac{\omega_2}{\omega_1 + \omega_2} \right) = t^2(\omega_1, \omega_2, \omega_1, \omega_2),$$

and (9) follows.

By a slight abuse of notation denote $t(a) := t(a, 1 - a, a, 1 - a)$. $t(a)$ is a twice continuously differentiable function. To find the maximizer of $t(a)^2$ we consider first the first-order condition with respect to a ,

$$\frac{\partial}{\partial a} t(a)^2 = 0. \tag{10}$$

Suppose $C \neq 0$, then this equation has two solutions,

$$a_1^* := \frac{E[YZ]}{E[YZ] - E[XZ]}$$

and

$$a_2^* := \frac{E[XZ]E[Z^2] - E[Z^2]E[XY]}{C}.$$

The first solution yields $t(a_1^*)^2 = 0$ and, therefore, cannot be the supremum. The second solution yields

$$t(a_2^*)^2 = n \frac{A}{B}.$$

By the Cauchy Schwarz inequality, $(E[XZ])^2 \leq E[Z^2]E[X^2]$, but equality is ruled out by Assumption 1(ii). Since $A \geq 0$ and $E[Y^2] > 0$, we have $B > 0$. It remains to show that this is indeed the supremum. For $\beta = 0$, the t-ratio equals zero and is independent of the weight a . Therefore, we can restrict attention to the case $\beta \neq 0$. First, consider the second derivative at the postulated maximizer:

$$\frac{\partial^2}{\partial a^2} t(a)^2 \Big|_{a=a_2^*} = -\frac{2E[Y^2]C^4}{AB^2} < 0.$$

The inequality follows because $A > 0$ whenever $\beta \neq 0$, otherwise $\text{Var}(Z - \lambda X) = 0$ for $\lambda = E(ZY)/E(XY)$ which is ruled out by Assumption 1(ii). Also, $B > 0$, $C \neq 0$, and Assumption 1(i) guarantees that $E(Y^2) > 0$. Therefore, a_2^* is a local maximizer. Since the t-ratio is a continuously differentiable function, a_2^* is also a global maximizer of the squared t-ratio as long as the squared t-ratio's limit as $a \rightarrow \pm\infty$ is smaller than (or equal to) its value at a_2^* . One can show that the two limits are equal and that they are smaller than (or equal to) the squared t-ratio at a_2^* if the following condition holds:

$$-\frac{E[Y^2]C^2}{BD} \geq 0 \quad (11)$$

where

$$D := (E[Y(X - Z)])^2 - E[Y^2]E[(X - Z)^2].$$

By the Cauchy-Schwarz inequality, we have $D \leq 0$, but equality is ruled out by Assumption 1(iii), so $D < 0$. $D < 0$ and $B > 0$ then imply that (11) holds and we thus have shown that a_2^* is indeed the global maximizer of the squared t-ratio when $C \neq 0$. We now turn to the case $C = 0$. In this case, the first-order condition (10) has only one solution,

$$a_3^* := \frac{E(ZY) [E(XZ)E(ZY) - E(XY)E(Z^2)]}{E(XY)(E(XY) - 2E(ZY))E(Z^2) + (2(XZ) - E(X^2))(EZY)^2},$$

which, when substituted back into the t-ratio yields $t(a_3^*) = 0$ and therefore cannot be a maximizer. Therefore, the maximum of the t-ratio is

$$\lim_{a \rightarrow \infty} t(a)^2 = \lim_{a \rightarrow -\infty} t(a)^2 = -\frac{(E[(X - Z)Y])^2}{D} > 0$$

which is also equal to $t(a_2^*)^2$ because $a_2^* = \infty$ when $C = 0$. In conclusion, a_2^* is the global maximizer of the squared t-ratio for any value of C . The expression for $t(a_2^*)$ follows simply by evaluating the t-ratio at the global maximizer and some straight-forward algebra. Q.E.D.

Proof of Theorem 2. The t-ratio can be written as

$$\bar{t}(\omega) = \frac{\tilde{\beta}(\omega)}{\bar{\sigma}(\omega)} = \frac{E[\tilde{W}(\omega_1, \omega_2)\tilde{Y}]}{\sqrt{E[\tilde{\varepsilon}(\omega)^2\tilde{W}(\omega_1, \omega_2)^2]}}.$$

Define $\tilde{\omega} = \left(\frac{\omega_1}{\omega_1 + \omega_2}, \frac{\omega_2}{\omega_1 + \omega_2}, \frac{\omega_3}{\omega_3 + \omega_4}, \frac{\omega_4}{\omega_3 + \omega_4} \right)$. Then,

$$\begin{aligned}
\tilde{\varepsilon}(\tilde{\omega}) &= \tilde{Y} - \tilde{\beta}(\tilde{\omega})\tilde{W} \left(\frac{\omega_3}{\omega_3 + \omega_4}, \frac{\omega_4}{\omega_3 + \omega_4} \right) \\
&= \tilde{Y} - \frac{E \left[\frac{\tilde{W}(\omega_1, \omega_2)}{\omega_1 + \omega_2} \tilde{Y} \right]}{E \left[\frac{\tilde{W}(\omega_1, \omega_2)}{\omega_1 + \omega_2} \frac{\tilde{W}(\omega_3, \omega_4)}{\omega_3 + \omega_4} \right]} \frac{\tilde{W}(\omega_3, \omega_4)}{\omega_3 + \omega_4} \\
&= \tilde{Y} - \tilde{\beta}(\omega)\tilde{W}(\omega_3, \omega_4) \\
&= \tilde{\varepsilon}(\omega)
\end{aligned}$$

and therefore it is obvious that

$$\bar{t}(\tilde{\omega}) = \frac{\frac{E[\tilde{W}(\omega_1, \omega_2)\tilde{Y}]}{\omega_1 + \omega_2}}{\sqrt{E \left[\tilde{\varepsilon}(\omega)^2 \left[\frac{\tilde{W}(\omega_1, \omega_2)}{\omega_1 + \omega_2} \right]^2 \right]}} = \frac{E[\tilde{W}(\omega_1, \omega_2)\tilde{Y}]}{\sqrt{E[\tilde{\varepsilon}(\omega)^2 \tilde{W}(\omega_1, \omega_2)^2]}} = \bar{t}(\omega).$$

Define that $a_1 := \frac{\omega_1}{\omega_1 + \omega_2}$ and $a_2 := \frac{\omega_3}{\omega_3 + \omega_4}$. Then,

$$\bar{t}(\omega) = \bar{t}(a_1, 1 - a_1, a_2, 1 - a_2).$$

Q.E.D.

The following lemma is a slightly modified version of results in [Black, Berger, and Scott \(2000\)](#):

Lemma 2. 1. Under Assumptions 2, 3, and 5, if $\beta \geq 0$,

$$\max_{a=0,1} \beta_{OLS}(a) \leq \beta, \quad (12)$$

and, if $\beta \leq 0$, then

$$\beta \leq \min_{a=0,1} \beta_{OLS}(a). \quad (13)$$

2. Under Assumptions 2, 3, and 6, if $\beta \geq 0$,

$$\beta \leq \min_{a=0,1} \beta_{OLS-INV}(a), \quad (14)$$

and, if $\beta \leq 0$, then

$$\max_{a=0,1} \beta_{OLS-INV}(a) \leq \beta. \quad (15)$$

3. Under Assumptions 2 and 4, if $\beta \geq 0$,

$$\beta \leq \min_{a=0,1} \beta_{IV}(a, 1-a), \quad (16)$$

and, if $\beta \leq 0$, then

$$\max_{a=0,1} \beta_{IV}(a, 1-a) \leq \beta. \quad (17)$$

Proof. First, notice that by Assumption 2,

$$\beta_{OLS}(a) = \beta \cdot \frac{E[X^*W(a)]}{E[X^*W(a)] + E[U(a)W(a)]} \quad (18)$$

where $U(a) := aU + (1-a)V$. By Assumptions 3 and 5, $E[X^*W(a)] > 0$ and $E[U(a)W(a)] \geq 0$ for $a = 0, 1$, so

$$\frac{E[X^*W(a)]}{E[X^*W(a)] + E[U(a)W(a)]} \leq 1$$

and the lower bound in (14) and the upper bound in (15) follow. Now, notice that Assumption 2 also implies

$$\beta_{OLS-INV}(a) = \frac{\beta^2 E[(X^*)^2] + E[\varepsilon^2]}{\beta E[X^*W(a)]} \quad (19)$$

By Assumptions 3 and 6, if $\beta \geq 0$, then

$$\beta_{OLS-INV}(a) \geq \frac{\beta^2 E[(X^*)^2]}{\beta E[X^*W(a)]} \geq \beta,$$

yielding the upper bound in (14). The lower bound in (15) for $\beta \leq 0$ follows from the same argument by reversing the above inequalities.

Now, we show that the IV bounds hold. Notice that by Assumption 2,

$$\beta_{IV}(a, 1-a) = \beta \cdot \frac{E[X^*W(1-a)]}{E[W(a)W(1-a)]}. \quad (20)$$

Assumption 4 then implies that, for $a = 0, 1$, the numerator and denominator are both positive and the latter is not larger than the former, so the IV bounds (16) and (17) follow. Q.E.D.

Proof of Theorem 3. To simplify the presentation we use the notation σ_{AB} for $E[AB]$ and σ_A^2 for $E[A^2]$, where A and B are random variables. Consider the first statement of

the theorem. First, notice that the OLS bound holds for $a \in \{0, 1\}$ by Lemma 2. From the proof of Lemma 2, we know that $\beta_{OLS}(a) \leq \beta$ holds if, and only if,

$$\frac{E[X^*W(a)]}{E[X^*W(a)] + E[U(a)W(a)]} \leq 1. \quad (21)$$

To show that this inequality holds only for $a \in \{0, 1\}$ and not for any other value of a , we assume that $a \notin \{0, 1\}$ and then construct a data-generating process characterized by

$$(\sigma_{\epsilon X^*}, \sigma_{\epsilon U}, \sigma_{\epsilon V}, \sigma_{X^*}^2, \sigma_U^2, \sigma_V^2, \sigma_{UV}, \sigma_{X^*U}, \sigma_{X^*V})$$

which satisfies all assumptions of the theorem, but violates (21).

Suppose $0 < a < 1$. Let $\epsilon > 0$ and $\sigma_U^2 > 0$. Set $\sigma_{UV} = \sigma_{X^*U} = \sigma_{\epsilon X^*} = \sigma_{\epsilon U} = \sigma_{\epsilon V} = 0$, $\sigma_{X^*}^2 = \sigma_V^2 = a\sigma_U^2/(1-a) + \epsilon$, and $\sigma_{X^*V} = -\sigma_V^2 + a\epsilon/2$. Then Assumption 2 holds. Since $\sigma_{X^*V} + \sigma_{UV} \leq 0$, $\sigma_{X^*U} + \sigma_{UV} \leq 0$, and

$$E[XZ] = \sigma_{X^*}^2 + \sigma_{X^*U} + \sigma_{X^*V} + \sigma_{UV} = \frac{a\epsilon}{2} > 0,$$

Assumption 4 holds, which implies Assumption 3. Assumption 5 is satisfied by definition of the above quantities. We also have

$$E[X^*W(a)] = aE[X^*X] + (1-a)E[X^*Z] > 0.$$

By the definition of σ_V^2 ,

$$\sigma_V^2 = \frac{a^2\sigma_U^2 + (1-a)^2\sigma_V^2}{1-a} + a\epsilon,$$

so that

$$(1-a)\sigma_{X^*V} < -(a^2\sigma_U^2 + (1-a)^2\sigma_V^2)$$

and thus

$$E[U(a)W(a)] = a\sigma_U^2 + (1-a)^2\sigma_V^2 + (1-a)\sigma_{X^*V} < 0.$$

Therefore, (21) is violated.

Suppose $a < 0$. Let $\epsilon > 0$, $\delta > 0$, and $\sigma_U^2 > 0$. If $a < -2$, then pick δ small enough so that $\delta \leq -4\epsilon/(2+a)$. Set $\sigma_{X^*U} = \sigma_{\epsilon X^*} = \sigma_{\epsilon U} = \sigma_{\epsilon V} = 0$, $\sigma_{UV} = -a\sigma_U^2/(2(1-a)) + \epsilon$, $\sigma_V^2 = 2\sigma_{UV} + a\sigma_U^2/(1-a) - \delta$, $\sigma_{X^*V} = -\sigma_V^2 - \delta a/2$, and $\sigma_{X^*}^2 = \sigma_V^2(1-a)$.

Then Assumption 2 holds. Assumption 3 holds because $E[XX^*] = \sigma_{X^*}^2 + \sigma_{X^*U} = \sigma_{X^*}^2 > 0$ and

$$E[ZX^*] = \sigma_{X^*}^2 + \sigma_{X^*V} = \sigma_V^2(1-a) - \sigma_V^2 - \frac{\delta a}{2} > 0.$$

Assumption 5 is satisfied by definition of the above quantities. We also have

$$E[X^*W(a)] = \sigma_{X^*}^2 + a\sigma_{X^*U} + (1-a)\sigma_{X^*V} = \sigma_{X^*}^2 - (1-a)\sigma_V^2 - \frac{(1-a)a\delta}{2} = -\frac{(1-a)a\delta}{2} > 0.$$

By the definition of σ_V^2 ,

$$\sigma_V^2 = \frac{a^2\sigma_U^2 + (1-a)^2\sigma_V^2 + 2a(1-a)\sigma_{UV}}{1-a} - a\delta,$$

so that

$$(1-a)\sigma_{X^*V} < -(a^2\sigma_U^2 + (1-a)^2\sigma_V^2 + 2a(1-a)\sigma_{UV})$$

and thus

$$E[U(a)W(a)] = a\sigma_U^2 + (1-a)^2\sigma_V^2 + 2a(1-a)\sigma_{UV} + (1-a)\sigma_{X^*V} < 0.$$

Therefore, (21) is violated.

In the case $a > 1$, we can construct a data-generating process violating (21) in a similar fashion as in the case $a < 0$.

Now, consider the second statement of the theorem. From the proof of Lemma 2, we know that the inverse OLS bound holds if, and only if,

$$\frac{\beta^2 E[(X^*)^2] + E[\varepsilon^2]}{\beta E[X^*W(a)]} \geq \beta. \quad (22)$$

This inequality holds for all $a \in [0, 1]$ by the same reasoning as in the proof of Lemma 2 because $E[X^*W(a)] > E[(X^*)^2]$ for all $a \in [0, 1]$.

To show that (22) does not hold for any a outside the interval $[0, 1]$, we construct a data-generating process that satisfies all assumptions of the theorem but violates (22). If $a < 0$, pick $\sigma_{X^*}^2 > 0$, $\sigma_{X^*U} \leq 0$, $\sigma_{X^*V} \leq 0$, such that $E[X^*X] > (1 - 1/a)E[X^*Z]$, $\sigma_{X^*}^2 + \sigma_{X^*U} > 0$, and $\sigma_{X^*}^2 + \sigma_{X^*V} > 0$. If $a > 1$, then pick $\sigma_{X^*}^2 > 0$, $\sigma_{X^*U} \leq 0$, $\sigma_{X^*V} \leq 0$ such that $E[X^*X] < (1 - 1/a)E[X^*Z]$, $\sigma_{X^*}^2 + \sigma_{X^*U} > 0$, and $\sigma_{X^*}^2 + \sigma_{X^*V} > 0$. In both cases, Assumptions 3 and 6 hold and Assumption 2 can, as above, be satisfied by simply choosing the three covariances to be zero. However,

$$E[X^*W(a)] = aE[XX^*] + (1-a)E[ZX^*] < 0$$

and thus (22) is violated.

Finally, consider the third statement of the theorem. The IV bound holds if, and only if,

$$\frac{E[X^*W(1-a)]}{E[W(a)W(1-a)]} \geq 1. \quad (23)$$

Suppose $0 < a < 1$. Set $\sigma_{UV} = \sigma_{X^*V} = \sigma_{X^*U} = \sigma_{\varepsilon X^*} = \sigma_{\varepsilon U} = \sigma_{\varepsilon V} = 0$ and let $\sigma_U^2, \sigma_V^2, \sigma_{X^*}^2$ any positive values. Then Assumptions 2 and 4 hold. However,

$$E[X^*W(1-a)] = \sigma_{X^*}^2 + E[X^*U(1-a)] = \sigma_{X^*}^2 > 0$$

and

$$E[U(a)W(1-a)] = a(1-a)(\sigma_U^2 + \sigma_V^2) > 0,$$

which means the bound (23) is violated.

Suppose $a < 0$. Set $\sigma_{UV} = \sigma_{X^*V} = \sigma_{\varepsilon X^*} = \sigma_{\varepsilon U} = \sigma_{\varepsilon V} = 0$. Pick $\sigma_U^2, \sigma_V^2 > 0$ and σ_{X^*U} small enough so it satisfies $\sigma_{X^*U} < -(1-a)(\sigma_U^2 + \sigma_V^2)$. Choose $\sigma_{X^*}^2 > -(1-a)\sigma_{X^*U}$. Then Assumptions 2 and 4 are satisfied because

$$E[XZ] > a\sigma_{X^*U} > 0$$

and $\sigma_{X^*U} + \sigma_{UV} \leq 0$ and $\sigma_{X^*V} + \sigma_{UV} \leq 0$. However,

$$E[X^*W(1-a)] = \sigma_{X^*}^2 + (1-a)\sigma_{X^*U} + a\sigma_{X^*V} > 0$$

and

$$E[U(a)W(1-a)] = a(1-a)(\sigma_U^2 + \sigma_V^2) + a\sigma_{X^*U} > 0,$$

which means, again, the bound (23) is violated. In the case $a > 1$, such a violation of (23) can be created in a similar fashion. Q.E.D.

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