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Understanding the Effect of Measurement Error on Quantile Regressions

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Abstract. The impact of measurement error in explanatory variables on quantile regression functions is investigated using a small variance approximation. The approximation shows how the error contaminated and error free quantile regression functions are related. A key factor is the distribution of the error free explanatory variable. Exact calculations probe the accuracy of the approximation. The order of the approximation error is unchanged if the density of the error free explanatory variable is replaced by the density of the error contaminated explanatory variable which is easily estimated. It is then possible to use the approximation to investigate the sensitivity of estimates to varying amounts of measurement error.

Keywords: measurement error, parameter approximations, quantile regression.

JEL Classification: C13, C14, C21

1. Introduction

Since the seminal paper of Koenker and Bassett (1978) there has been substantial development of estimation methods and algorithms for quantile regression functions (QRF), and gains in understanding of the properties of QRF estimators. With increasing interest in econometrics in variation in response amongst individuals, and with the way in which the distribution of responses is affected by covariates, the

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use of quantile regression estimation procedures has become widespread in applied econometric work.

Despite many advances in QRF estimation and inference methods, and the many applications of quantile regression methods, some of the econometric issues given substantial attention in the study of mean regressions have received little attention in the context of quantile regressions. One of these is *measurement error in explanatory varaibles*, a pervasive feature of econometric data, and likely a feature in many applications.

Covariate measurement error causes many and subtle changes to conditional distributions, potentially attenuating mean regression functions, increasing dispersion, introducing heteroskedasticity in homoskedastic error free models and modifying the form of heteroskedasticity when it is present in an error free model. There are significant effects on quantile regression functions.

This paper develops results that improve understanding of these effects. It does this by developing approximations to QRFs in which covariate values are contaminated by measurement error. These are Taylor series expansions around a point at which measurement error is absent. The approximations reveal the first order effect of covariate measurement error on QRFs and lead to a procedure for investigating the magnitude of this effect when there are neither instrumental variables nor repeated measurements. The probability distribution of measurement errors does not feature in the approximation, only its variance. The major first order influence of measurement error on QRFs is found to be the shape of the distribution of error free covariates and the way this interacts with the shape of the error free QRF.

There has been little attention paid to measurement error in the context of quantile regression. Measurement error oriented texts such as Fuller (1987) and Carroll, Ruppert, Stefanski and Crainiceanu (2006) have no discussion of quantile regression. Koenker's Econometric Society Monograph on quantile regression, Koenker (2005), has no discussion of covariate measurement error.

This paper considers error free QRFs for a response Y, conditioned on X, and error contaminated QRFs for Y, conditioned on $Z = X + V$ where V is distributed independently of X, and of Y given X. Data on Y and the error contaminated Z provide information about the way in which the conditional quantiles of Y given Z vary with Z. Nonparametric quantile regression methods can give detailed information about this variation. But this bears only indirectly on the way in which quantiles of Y conditional on error free X vary with X. In most applications this is what is of interest because economic theory will be informative about relationships between error free variates, and policy interventions, whose impact on the distribution of Y is of interest, will alter values of error free covariates.

The first part of the paper gives results that improve understanding of the relationship between error contaminated and error free QRFs. This helps interpret the results of QRF estimation when measurement error is believed to be present. It helps explain differences in estimated QRFs using data sets with different amounts of measurement error. In cases where a functional form of an error free QRF is imposed it is informative about the misspecification that is committed when error contaminated data is used.

The focus of the second part of the paper is on problems in which error free QRFs are parametrically specified. The possibility of using information on the relationship between error free and error contaminated QRFs to retrieve information about the values of parameters of error free QRFs is investigated. In some circumstances this is not possible because the error free QRF cannot be identified from knowledge of the form of error free QRFs because measurement error induces no change in that form. The model in which Y, X and V are jointly normally distributed is a leading example. Here measurement error changes the separation and slope of QRFs but they remain linear. But in many other cases identification is possible, as pointed out in Reiersøl (1950).

The analysis of Reiersøl (1950) is extended in Schennach and Hu (2013), SH13.

SH13 considers additive error models for an outcome with parallel QRFs and shows that outside a small class of models for dependence of an outcome on an error free covariate the error free QRF can be identified. The approximations developed here apply to a wider class of models which does not require an additive error. The class includes the SH13 model. The approximations of this paper could be used to decide whether estimation via SH13 is worthwhile if an additive error model were being cosidered, and generally to gain understanding of the potential effect of measurement error in the context of a specific analysis of data.

The procedure we propose provides a form of sensitivity analysis. It provides a partial answer to the following questions.

> Were the error free QRF to be of a hypothesised form and covariate measurement error to be present, what are the likely values of the parameters of the error free QRFs? How does our view of this change as the amount of measurement error allowed for increases? Are some parameters substantially affected by measurement error relative to others?

An exact answer to these questions requires a case by case analysis. The exact impact of measurement error on mean regressions can be derived in explicit form only in a few special cases[1]. Outside these cases, and in almost all interesting cases for QRFs, the exact impact of covariate measurement error can only be obtained by numerical calculation. Such calculations do not give insight into the generic effects of covariate measurement error and they do not provide a link between the effects of measurement error and easily grasped features of the error free QRF and the distributions of covariates and measurement error.

This paper provides a partial resolution of this problem by providing an *approximation* to an error contaminated QRF expressed in terms of functionals of the error free QRF and the density of either the error free or the error contaminated covari-

[1] A leading example is the model in which Y (with arbitrary distribution) has *polynomial* regression on error free X and additive independent measurement error is normally distributed, see Chesher (1998a).

ates. The approximation is developed using small parameter (variance) approximation methods and extends the results of Chesher (1991) to QRFs.

Section 2 gives the approximation to error contaminated QRFs. Details of the derivation are given in an appendix. The insights into the generic effects of measurement error on QRFs provided by the approximation are discussed in Section 3 where some leading special cases are examined.

Section 4 reports an investigation into the accuracy of the approximation in a rich class of models with a single error contaminated covariate. An error free covariate (X) and independently distributed measurement error (V) are given exponential power distributions[2]. The conditional distribution of the response given X (independent of V) is also specified as a member of the exponential power family with location parameter depending upon X and with scale and shape parameters independent of X. The exact error free (conditional on X) and error contaminated (conditional on $Z = X + V$) QRFs are calculated and the approximation developed in Section 2 is calculated. For quite large amounts of measurement error the approximations are acceptably accurate.

Sections 5 considers one possible use of the approximation. It investigates the use of the approximate QRF to extract information about the error free QRF from error contaminated data. The results of Section 2 show that the approximate error contaminated QRF is determined by the error free QRF and derivatives of it, whose form is known once the error free QRF is specified, and by a functional of the density of the error contaminated covariate. This density can be estimated using realizations of the error contaminated covariate. Therefore, given a parametric form for the error free QRF, a parametric approximate error contaminated QRF can be specified and estimated. When identification permits, estimates of parameters of the error free QRF can be retrieved.

The performance of this procedure is investigated in Monte Carlo experiments. In

[2] A random variable with an exponential power distribution has density function proportional to $\exp(-\lambda|w-\mu|^{\frac{2}{1+b}})$. Normal ($b = 0$) and Laplace ($b = 1$) distributions are leading special cases.

practice one would likely want to use this procedure to perform a sensitivity analysis. Thus one can ask: given a specified form for an error free QRF, how are my views about its parameters changed as I consider the possibility of there being more or less measurement error. An alternative procedure suitable when there is no parametric specification of the error free QRF is proposed. Section 6 concludes.

This Section concludes with a brief outline of the few available results on QRF estimation with covariate measurement error.

Brown (1982), studying robust estimation in errors-in-variables models, proposes a modified LAD estimator which can be regarded as an estimator of the slope of the median regression function with scalar error contaminated covariate, but rejects the estimator as unsatisfactory. He and Liang (2000) propose a consistent estimator of the slope of linear error free QRFs based on minimising the sum across observations of the "check" functions

$$\rho_\tau(r) = r \times (\tau - 1_{[r<0]})$$

applied to orthogonal residuals, r. They assume that the joint distribution of the response error and the covariate measurement error is spherically symmetric. Hu and Schennach (2008) and Schennach (2008) develop identification and estimation results using instrumental variables or repeated measurements. Wei and Carroll (2009) propose an estimation procedure for linear quantile regression models in situations in which one can estimate the distribution of the error free covariate conditional on its error contaminated version. Montes-Rojas (2011) applies the misspecification analysis of Angrist, Chernozhukov and Fernandez-Val (2006) to obtain a formula for the probability limit of the QRF estimator in a linear model with parallel QRFs when there is a measurement error contaminated covariate. Schennach and Hu (2013), considering an additive error model with parallel QRFs, give conditions on the functional form of an error free QRF under which it can be identified from knowledge of the distribution of the outcome variable and the error contaminated covariate. Shang (2012) and Shang, Vanlwaarden and Bettebenner (2014) apply the SIMEX method of Carroll and Stefanski (1994) to correct estimates of the distribution of current student test scores conditional on past scores. Firpo, Galvao and Song (2017) develop an estimator of the coefficients in an error free linear quantile regression function when repeated error contaminated measurements of the scalar error free explanatory variable are available. Hausman, Luo and Palmer (2014) consider the impact on QRF estimation of measurement error in the response variable.

2. THE APPROXIMATE EFFECT OF MEASUREMENT ERROR

2.1. Error free and error contaminated QRFs. Consider a scalar response Y, continuously distributed given k element vector X, and let $F_{Y|X}(y|x)$ be the conditional distribution function of Y given $X = x$, as follows.

$$F_{Y|X}(y|x) = P[Y \leq y|X = x]$$

Let $Z = X+V$ where $V = \Psi U$, U and X are independently distributed and $E[U] = 0$, $Var[U] = I$. The matrix Ψ is lower triangular and $\Psi\Psi' = \Sigma$ so that $Var[V] = \Sigma = [\sigma_{ij}]$.

The conditional τ-QRFs, $Q_X(\tau, x)$, in which conditioning is on *error free* X, and $Q_Z(\tau, z)$, in which conditioning is on *error contaminated* Z, are defined by the following implicit equations.

$$\begin{aligned} F_{Y|X}(Q_X(\tau, x)|x) &= \tau \\ F_{Y|Z}(Q_Z(\tau, z)|z) &= \tau \end{aligned}$$

2.2. Approximate error contaminated QRFs. We seek an approximation to the error contaminated τ-QRF, $Q_Z(\tau, z)$. This is a functional of the conditional distribution function of Y given X and the marginal distribution functions of U and

X and depends upon τ and Σ, a relationship we express as follows.

$$Q_Z(\tau,\cdot) = \mathcal{F}(F_{Y|X}, F_X, F_U; \tau, \Sigma) \tag{1}$$

Note that the error free QRF is got by setting $\Sigma = 0$.

$$Q_X(\tau,\cdot) = \mathcal{F}(F_{Y|X}, F_X, F_U; \tau, 0)$$

The approximation to the error contaminated QRF is given in equation (6) below, to which those not interested in the method of derivation can proceed directly.

The approximation is obtained by considering a first order Taylor series type approximation to $Q_Z(\tau,\cdot)$ defined in (1) around $\Sigma = 0$. This takes the following form[3]

$$Q_Z(\tau,\cdot) = Q_X(\tau,\cdot) + \sum_{i,j} \sigma_{ij} \frac{\partial}{\partial \sigma_{ij}} \, Q_Z(\tau,\cdot)|_{\Sigma=0} + o(\Sigma) \tag{2}$$

where σ_{ij} is the (i,j) element of the measurement error variance matrix Σ. The leading term is just the τ-QRF of Y given error free X.

The following approximation to the conditional distribution function $F_{Y|Z}(y|z)$ derived in Chesher (1991) is used. Here conditioning is on error contaminated Z.

$$F_{Y|Z}(y|z) = \tilde{F}_{Y|Z}(y|z) + o(\Sigma) \tag{3}$$

where

$$\tilde{F}_{Y|Z}(y|z) = F_{Y|X}(y|z) + \sum_{i,j} \sigma_{ij} \left(F^i_{Y|X}(y|z) g^j_X(z) + \frac{1}{2} F^{ij}_{Y|X}(y|z) \right). \tag{4}$$

Here superscripts i, j indicate differentiation with respect to the ith and jth condi-

[3] Here and later unless indicated $\sum_{i,j}$ indicates double summation over i and j both from 1 to k. A term described as $o(\Sigma)$ has the property that

$$\lim_{\omega \to 0} \frac{o(\Sigma)}{\omega} = 0$$

where $\omega = \text{trace}(\Sigma)$.

tioning arguments, for example

$$F^{ij}_{Y|X}(y|z) = \left.\frac{\partial^2}{\partial x_i \partial x_j} F_{Y|X}(y|x)\right|_{x=z}.$$

The function $g_X(\cdot)$, which plays a key role in what follows, is the log probability density function of X,

$$g_X(z) \equiv \log f_X(z)$$

with derivatives as follows.

$$g^j_X(z) \equiv \left.\frac{\partial}{\partial x_j} g_X(x)\right|_{x=z}$$

For the approximation to have an error of the stated order we require the absolute third own and cross moments of U to be finite, the existence of bounded third own and cross derivatives of $F_{Y|X}(y|x)$ with respect to x, and that X has a continuous distribution with twice differentiable density function and support on $\Re^k$. The approximation (4) to the distribution function does not require Y to be continuously distributed[4] given X.

For the moment let Q_Z be shorthand for $Q_Z(\tau, z)$. Since $F_{Y|Z}(Q_Z|z) = \tau$ we have, from (3),

$$\tilde{F}_{Y|Z}(Q_Z|z) = \tau + o(\Sigma),$$

that is:

$$F_{Y|X}(Q_Z|z) + \sum_{i,j} \sigma_{ij} \left(F^i_{Y|X}(Q_Z|z) g^j_X(z) + \frac{1}{2} F^{ij}_{Y|X}(Q_Z|z) \right) = \tau + o(\Sigma).$$

Considering variation in Q_Z and Σ and taking differentials gives

$$F^Y_{Y|X}(Q_Z|z) dQ_Z + \sum_{i,j} d\sigma_{ij} \left(F^i_{Y|X}(Q_Z|z) g^j_X(z) + \frac{1}{2} F^{ij}_{Y|X}(Q_Z|z) \right) + O(\Sigma) = o(\Sigma),$$

[4] In its application to QRFs we do assume a continuous distribution for Y with strictly increasing distrbution function,

where the superscript "Y" denotes differentiation with respect to the response variable, that is:

$$F^Y_{Y|X}(Q_Z|z) \equiv \frac{\partial}{\partial y} F_{Y|X}(y|z)\bigg|_{y=Q_Z} = f_{Y|X}(Q_Z|z)$$

which is the conditional density of Y at the τ-quantile under consideration. Setting $\Sigma = 0$, yields the required derivatives,

$$\frac{\partial Q_Z}{\partial \sigma_{ij}}\bigg|_{\Sigma=0} = -\frac{F^i_{Y|X}(Q_Z|z)g^j_X(z) + \frac{1}{2}F^{ij}_{Y|X}(Q_Z|z)}{F^Y_{Y|X}(Q_Z|z)}$$

and, plugging in to (4) there is the following approximation.

$$Q_Z(\tau, z) = Q_X(\tau, z) - \sum_{i,j} \sigma_{ij} \frac{F^i_{Y|X}(Q_Z|z)g^j_X(z) + \frac{1}{2}F^{ij}_{Y|X}(Q_Z|z)}{F^Y_{Y|X}(Q_Z|z)} + o(\Sigma) \qquad (5)$$

The approximation is more easily interpreted, when expressed in terms of the error free QRF and its derivatives

$$\begin{aligned} Q^\tau_X(\tau, z) &= \frac{\partial}{\partial \tau} Q_X(\tau, x)\bigg|_{x=z} \\ Q^i_X(\tau, z) &= \frac{\partial}{\partial x_i} Q_X(\tau, x)\bigg|_{x=z} \end{aligned}$$

and so forth, as follows. Details of the derivation of this expression are given in Appendix 1.

$$\begin{aligned} Q_Z(\tau, z) = \; & Q_X(\tau, z) + \sum_{i,j} \sigma_{ij} \left(Q^i_X(\tau, z) g^j_X(z) + \frac{1}{2} Q^{ij}_X(\tau, z) \right) \\ & - \frac{1}{2} \frac{1}{Q^\tau_X(\tau, z)} \sum_{i,j} \sigma_{ij} \left(Q^{\tau i}_X(\tau, z) Q^j_X(\tau, z) + Q^{\tau j}_X(\tau, z) Q^i_X(\tau, z) \right) \\ & + \frac{1}{2} \frac{Q^{\tau\tau}_X(\tau, z)}{Q^\tau_X(\tau, z)^2} \sum_{i,j} \sigma_{ij} Q^i_X(\tau, z) Q^j_X(\tau, z) + o(\Sigma) \qquad (6) \end{aligned}$$

2.3. Discussion. Section 3 provides interpretation of the terms in this approximation and considers some leading special cases. First it is worth noting that there are elements of generality that may not be obvious at first sight.

Non-additive measurement error. The approximation has been developed for the case of additive measurement error, but we have allowed the error free QRF to be nonlinear, so some other interesting cases can be easily obtained by considering transformations of the covariates. For example[5] consider a scalar covariate X and let

$$Z = \lambda^{-1}(\lambda(X) + \lambda(V))$$

where $\lambda(\cdot)$ is a strictly monotonic function. Additive and multiplicative measurement error arise when $\lambda(\cdot)$ is respectively the identity function and the logarithmic function. The approximation (6) for additive measurement error applies when the error free QRF is expressed as a function of $\lambda(X)$. Then $g_X(\cdot)$ must be regarded as the log density of $\lambda(X)$. The result is easily "unbundled" to give an approximation in terms of an error free QRF written as a function of X and the log density of X. Of course the resulting approximation will involve the function $\lambda(\cdot)$ and its derivatives[6].

Error free covariates. We have proceeded as if all elements of X are error contaminated, but in many leading cases of interest we may expect measurement error to be a serious issue for only one covariate. For example in considering household demand we may be confident in the accuracy of measures of household composition but suspect measurement error in household income. The approximation (6) is easily applied to such cases by setting elements of Σ to zero. Note that in this case, with X_F and X_C denoting respectively error free and error contaminated covariates, the log density derivative $g_X^j(z)$ that appears in (6) becomes the derivative of the log *conditional* density of X_C given X_F with respect to elements of X_C.

[5] I am grateful to Christian Schluter for suggesting this generalised additive formulation.

[6] This is essentially the approach taken in Chesher and Schluter (2002) and in Chesher, Dumangane and Smith (2002) in studing the impact of measurement error on respectively inequality measures (e.g., the Gini coefficient) and duration analysis. In both cases multiplicative measurement error is the leading case of interest.

Alternative forms of the approximation. The log density derivatives $g_X^j(z)$ that appear in (6) can be replaced by derivatives of the log density of Z, $g_Z^j(z)$, without increasing the order of the approximation error. This is proved in Appendix 2. This substitution has two benefits. First, in models with normal measurement error it can result in increased accuracy[7]. Second, unlike $g_X^j(z)$, the function $g_Z^j(z)$ can be estimated - using realizations of error contaminated Z. With an estimate of $g_Z^j(z)$ and knowledge of the form of the error free QRF one then has information about all aspects of the dependence on z of the approximate error contaminated QRF, a point that is crucial to our proposed sensitivity analysis procedure.

3. Interpretation and special cases

First it is interesting to compare the quantile regression approximation (6) with the approximate mean regression function given in Chesher (1991). For error free and error contaminated mean regression functions respectively $R_X(x) \equiv E_{Y|X}[Y|X = x]$ and $R_Z(z) \equiv E_{Y|Z}[Y|Z = z]$, with error contaminated $Z = X+V$, this approximation is as follows.

$$R_Z(z) = R_X(z) + \sum_{i,j} \sigma_{ij} \left(R_X^i(z) g_X^j(z) + \frac{1}{2} R_X^{ij}(z) \right) + o(\Sigma) \qquad (7)$$

This has the same form as the first line of (6)[8].

The second and third lines in (6) capture (approximately) the variance and distributional shape distortions produced by measurement error. Most of the message contained in these approximations can be uncovered by considering the case in which there is just one covariate, which is the case considered now.

[7] When error free mean regressions are linear a substitution of this sort renders the approximation to mean regressions exact, Chesher (1998a).

[8] It also has the same form as (4) because the conditional distribution function $F_{Y|X}(y|x)$ *is* a regression function, namely the regression of $1_{[Y \leq y]}$ on X. For the mean regression function approximation to have a remainder term that is $o(\Sigma)$ the third order derivatives of the error free regression function are required to be bounded.

3.1. Attenuation and curvature damping. Let superscript "x" denote differentiation with respect to the single covariate and write the scalar measurement error variance as σ^2. When there is one covariate (6) simplifies as follows.

$$\begin{aligned} Q_Z(\tau,z) \;=\; & Q_X(\tau,z) + \sigma^2 Q_X^x(\tau,z) g_X^x(z) + \frac{\sigma^2}{2} Q_X^{xx}(\tau,z) \\ & -\sigma^2 \frac{Q_X^{\tau x}(\tau,z) Q_X^x(\tau,z)}{Q_X^\tau(\tau,z)} \\ & +\frac{\sigma^2}{2} \frac{Q_X^{\tau\tau}(\tau,z) Q_X^x(\tau,z)^2}{Q_X^\tau(\tau,z)^2} + o(\sigma^2) \end{aligned} \tag{8}$$

The leading term is just the error free QRF with argument z. The next two terms completing the first line of (8) do not involve derivatives with respect to τ. These are QRF analogues of the only $O(\Sigma)$ terms in the mean regression approximation (7).

The term $g_X^x(z)$ is zero at every mode of the density of X. To the left (right) of each mode $g_X^x(z)$ is positive (negative). Consider x and τ where the error free QRF has a positive derivative. There the effect of the term $\sigma^2 Q_X^x(\tau,z) g_X^x(z)$ is to raise the error contaminated QRF relative to the error free QRF to the left of each mode of the density of X and to lower it to the right of each mode. This tends to "flatten" the QRF and is an expression of the *attenuating effect* of measurement error. There is the same attenuation effect where the error free QRF has a negative derivative.

The effect is clear to see when the error free QRF is linear and is illustrated for mean regression in Chesher (1991). Then $Q_X^x(\tau,z)$ is constant and the term $\frac{\sigma^2}{2} Q_X^{xx}(\tau,z)$ vanishes. When $g_X^x(z)$ is linear, which occurs when X is normally distributed, the approximate error contaminated QRF is linear, but otherwise the term $g_X^x(z)$ introduces *nonlinearity.* The nonlinearity induced by measurement error can be seen in Figures 1, 2 and 3 which show exact error free and error contaminated QRFs and approximations to the latter for a set up described in Section 4.2.

The opposite effect occurs at each antimode of the density of X. Near antimodes the error contaminated QRF is ampliated. The result is that when the distribution of X is multimodal the error contaminated QRF tends to move sinuously relative to the error free QRF.

The final term in the first line of (8) is present only when the error free QRF is nonlinear. It is positive where that QRF is strictly concave, tending to reduce the degree of concavity, and positive where the error free QRF is convex, tending to decrease the degree of convexity. The effect of this term is to dampen the curvature of the error contaminated QRF relative to the error free QRF.

The terms in the second and third lines of (8) are more complex and more easily understood in special cases. We first consider them in problems in which error free QRFs are parallel.

3.2. Parallel conditional quantiles. Consider parallel error free QRFs

$$Q_X(\tau, x) = a(\tau) + b(x)$$

which arise when Y is a location shift of a random variable W, the latter distributed independently of X, that is

$$Y = b(X) + W.$$

With $Q_W(\tau) = a(\tau)$ denoting the τ-quantile of W,

$$Q_X(\tau, x) = Q_W(\tau) + b(x).$$

In this case $Q_X^{\tau x}(\tau, z) = 0$ which *removes* the term in the second line of (8).

In this case, applying (8), the error contaminated quantile is approximately

$$Q_Z(\tau, z) = a(\tau) + b(z) + \sigma^2 b^x(z) g_X^x(z) + \frac{\sigma^2}{2} b^{xx}(z) + \frac{\sigma^2}{2} \frac{a^{\tau\tau}(\tau) b^x(z)^2}{a^\tau(\tau)^2} + o(\sigma^2) \quad (9)$$

where superscripts "x" and "τ" denote differentiation with respect to x and τ respectively. The following points are of interest[9].

[9] Where statements are made about some manifestation of measurement error being present or absent it should be taken to mean to the order of approximation considered in this analysis.

1. Even though the error free quantiles are parallel, the error contaminated quantiles are *not* in general parallel, because in the final term of (9) there are functions of z and τ which interact.

2. However if the error free quantile functions are *linear* the final term in (9) is a function of τ alone and measurement error does *not* destroy the parallel quantile property, though it may render quantile functions non-linear through the impact of the term $\sigma^2 b^x(z) g_X^x(z)$ in (9).

3. Regarding $a(\tau)$ as the quantile function of the random variable W, we have

$$\frac{a^{\tau\tau}(\tau)}{a^{\tau}(\tau)^2} = \frac{Q_W^{\tau\tau}(\tau)}{Q_W^{\tau}(\tau)^2} = -\left.\frac{\partial}{\partial w} \log f_W(w)\right|_{w=Q_W(\tau)} = -g_W^w(Q_W(\tau))$$

   where (A1.3) and (A1.5) of Appendix 1 have been used to obtain the final expression and $f_W(w)$ is the density function of W.

   (a) This term, and so the final term in (9), is zero at each mode (and antimode) of the density of W.

   (b) When the density of W is unimodal, this final term in (9) is negative for small τ and positive for large τ, and captures the impact of measurement error in increasing the dispersion of the conditional distribution of Y.

   (c) This dispersion increasing effect is larger for values of z at which $b^x(z)$ is large in magnitude and zero when $b^x(z)$ is zero. In the nonlinear quantile function case the variations with z in the sensitivity of $b(z)$ to z induce heteroskedasticity.

In summary, parallel nonlinear quantile regressions contaminated by measurement error become non-parallel, the effect being greater at covariate values at which error free QRFs are more nonlinear. The discussion of Section 2.3 implies that this effect will also be present in linear error free QRF problems when measurement error is not additive.

Error contaminated QRFs tend to be more widely separated then error free QRFs. This expansion effect is larger when the slope of the error free QRF is large in magnitude. It is larger for τ-QRFs for which τ corresponds to a quantile on a sharply increasing or decreasing part of the conditional density, in many cases this will be away from the mode of this distribution but in the main body of the distribution.

3.3. Non-parallel conditional quantiles. With non-parallel quantiles there is heteroskedasticity and/or conditional shape variation in the error free model and these are altered by the introduction of measurement error. This effect is captured in the term in (8) involving $Q_X^{\tau x}(\tau, z)$ which is nonzero only at points where quantile functions are non-parallel. Consider the simple case in which

$$Q_X(\tau, z) = a(\tau)c(x) + b(x)$$

which arises when

$$Y = b(X) + c(X)W$$

and W is independent of X with τ-quantile $Q_W(\tau) = a(\tau)$. The error free τ-quantile is ($c(x) \geq 0$ is assumed)

$$Q_X(\tau, x) = c(x)Q_W(\tau) + b(x).$$

The relevant term in (8) is

$$\frac{Q_X^{\tau x}(\tau, z)Q_X^{x}(\tau, z)}{Q_X^{\tau}(\tau, z)} = \frac{c^x(z)}{c(z)}\left(Q_W(\tau)c^x(z) + b^x(z)\right).$$

This term further modifies the τ-free part of the QRF adding the term $c^x(z)b^x(z)/c(z)$ and modifies the form of the covariate dependence of shape and dispersion.

4. ACCURACY OF THE APPROXIMATION

This section examines the accuracy of the approximation to error contaminated QRFs. Some of the results are obtained using numerical methods to calculate the exact error contaminated QRF, but first consider the fully Gaussian model in which the error contaminated QRF can be obtained in closed form. Here we find that the approximation is exact so far as capturing the dependence of the QRF on covariates is concerned.

4.1. Analytic results for a Gaussian model. Let (Y, X, V) be jointly normally distributed with Y given $X = x$ and $V = v$ distributed $N(x'\beta, \eta^2)$ and with

$$\begin{bmatrix} X \\ V \end{bmatrix} \sim N\left(\begin{bmatrix} \mu_X \\ 0 \end{bmatrix}, \begin{bmatrix} \Sigma_{XX} & 0 \\ 0 & \Sigma \end{bmatrix}\right).$$

The joint distribution of (Y, Z) is

$$\begin{bmatrix} Y \\ Z \end{bmatrix} \sim N\left(\begin{bmatrix} \mu_X'\beta \\ \mu_X \end{bmatrix}, \begin{bmatrix} \eta^2 + \beta'\Sigma_{XX}\beta & \beta'\Sigma_{XX} \\ \Sigma_{XX}\beta & \Sigma_{XX} + \Sigma \end{bmatrix}\right)$$

and the conditional distribution of Y given $Z = z$ is $N(\mu_{Y|Z}(z), \sigma^2_{Y|Z})$ where

$$\mu_{Y|Z}(z) \equiv \beta' \left(I - \Sigma_{XX}(\Sigma_{XX} + \Sigma)^{-1}\right) \mu_X + \beta'\Sigma_{XX}(\Sigma_{XX} + \Sigma)^{-1} z$$

$$\sigma^2_{Y|Z} \equiv \eta^2 + \beta'\Sigma_{XX}\left(\Sigma_{XX}^{-1} - (\Sigma_{XX} + \Sigma)^{-1}\right)\Sigma_{XX}\beta.$$

Let $Q_N(\tau)$ be the τ-quantile of a $N(0, 1)$ variate. It follows that the exact error free and error contaminated QRFs of Y are linear functions of respectively x and z, as follows.

$$Q_X(\tau, x) = x'\beta + \eta Q_N(\tau)$$

$$Q_Z(\tau, z) = \beta' \left(I - \Sigma_{XX}(\Sigma_{XX} + \Sigma)^{-1}\right) \mu_X + \beta'\Sigma_{XX}(\Sigma_{XX} + \Sigma)^{-1} z + a(\beta, \Sigma_{XX}, \Sigma, \eta) Q_N(\tau) \tag{10}$$

where

$$a(\beta, \Sigma_{XX}, \Sigma, \eta) = \sigma_{Y|Z} = \left(\eta^2 + \beta' \Sigma_{XX} \left(\Sigma_{XX}^{-1} - (\Sigma_{XX} + \Sigma)^{-1}\right) \Sigma_{XX}\beta\right)^{1/2}.$$

Consider the approximation (6) and the expression obtained if $g_Z^j(z)$ in place of $g_X^j(z)$ is employed, as suggested in Section 2.3. It is now shown that the coefficients on z in the approximate QRF calculated this way are identical to the coefficients on z in the exact QRF using error contaminated Z.

Since $Z \sim N(\mu_X, \Sigma_{XX} + \Sigma)$ the log density of Z is

$$g_Z(z) = A - \frac{1}{2}((z - \mu_X)' \left(\Sigma_{XX} + \Sigma\right)^{-1} (z - \mu_X))$$

where A does not depend on z. The z derivative of the log density is $-(\Sigma_{XX} + \Sigma)^{-1} (z - \mu_X)$. Plugging $g_Z^j(z)$ in place of $g_X^j(z)$ into (6), gives the following approximation to the error contaminated QRF.[10]

$$\tilde{Q}_Z(\tau, z) = \beta' \left(I - \Sigma_{XX}(\Sigma_{XX} + \Sigma)^{-1}\right) \mu_X + \beta' \Sigma_{XX}(\Sigma_{XX} + \Sigma)^{-1} z + \tilde{a}(\beta, \Sigma) Q_N(\tau) \tag{11}$$

$$\tilde{a}(\beta, \Sigma, \eta) = \eta + \frac{\beta' \Sigma \beta}{2\eta}$$

The first two terms in the approximation $\tilde{Q}_Z(\tau, z)$ given in (11) are identical to the first two terms in the exact expression for the error contaminated QRF, $Q_Z(\tau, z)$, given in (10) so the regression coefficients of this approximate QRF are the *same* as those of the exact QRF. Approximation error arises only in the intercept, and only because of error in $\tilde{a}$ as an approximation to a.

It follows that when distributions of unobservables are not far from Gaussian the approximation employed here can be quite accurate even when the measurement error variance is large.

[10] The term $Q_X^{ij}(\tau, z)$ on the first line of (6) and the term on the second line of (6) are zero. On the third line the summation is $\beta' \Sigma \beta$ and the multiplier of this term is simply $Q_N(\tau)/(2\eta)$. The remaining terms in the first line deliver the first two terms in (11) and the term $\eta Q_N(\tau)$ and hence the term η in $\tilde{a}(\beta, \Sigma, \eta)$.

4.2. Numerical calculations for exponential power distributions. It is difficult to find other cases in which exact error contaminated QRFs can be obtained in closed form so the accuracy of the approximation is now examined using numerical methods. Attention is confined to models with a single covariate.

A particular structure is defined and exact QRFs are calculated conditioning on an error free covariate and conditioning on an error contaminated covariate. Approximate QRFs conditioning on the error contaminated covariate are calculated and the exact error free and error contaminated and approximate error contaminated QRF are compared. The calculations are done using numerical integration procedures. These are not Monte Carlo experiments, rather exact calculations (within the bounds of computational accuracy) to show the difference between the error contaminated QRF and error free QRF and the quality of the approximation to the latter proposed here.

In the structures studied, Y is determined by a location shift model in which

$$Y = \beta_0 + \beta_1 X + \sigma_W W \tag{12}$$

$$Z = X + \sigma U \tag{13}$$

with Z an error contaminated measure of X. Unobserved mean zero W and $V \equiv \sigma U$ and X are mutually independently distributed with exponential power (EP) distributions[11] with shape parameters bw, bv and bx.

A random variable S with mean μ and variance λ^2 and an exponential power distribution with shape parameter $b \in (-1, 1)$ has the following probability density function.

$$f_S(s) = A \exp\left(-B\left|\frac{s-\mu}{\lambda}\right|^{\frac{2}{1+b}}\right)$$

The constants A and B are defined in Appendix 3. Setting b equal to 0 and 1 gives respectively normal and Laplace distributions. As $b \to -1$ the density approaches

[11] Box and Tiao (1973) give properties of EP distributions.

the uniform density function on $(\mu - \sqrt{3}\lambda, \mu + \sqrt{3}\lambda)$.

Let $Q_b(\tau)$ denote the τ-quantile[12] of a zero mean unit variance EP variate with shape parameter b. Then the error free QRF of Y is

$$Q_X(\tau, x) = \beta_0 + \beta_1 X + \sigma_W Q_{bw}(\tau). \tag{14}$$

To obtain the exact error contaminated QRF the conditional distribution function of Y given Z is calculated by numerical integration[13] and the value of the QRF at values of z is obtained using a Newton type method[14].

Figures 1, 2 and 3 show error free (dotted), exact error contaminated (solid) and approximate error contaminated (dashed) τ-QRFs when $\beta_0 = 0$, $\beta_1 = 1$, $\sigma_W = 1$, $\sigma_{XX} = 3$, and $\sigma^2 = 1$. At these settings R^2 in the error free mean regression is 0.75, the signal to noise ratio for the error contaminated covariate is 0.75, and for mean regression the attenuation of the error contaminated regression is 25%, that is $E[Y|Z = z] = 0.75z$ compared with $E[Y|X = x] = x$.

The graphs show τ-QRFs for $\tau \in \{0.5, 0.75, 0.9\}$. Figures 1, 2 and 3 are distinguished by the choice of shape parameter in the EP distribution for W, with bw equal to 0.5, 0 and -0.5 respectively. The variance of the error contaminated covariate is 4 and the graphs show QRFs for $z \in [-4, 4]$, that is ± 2 standard deviations around the mean.

In each 3×3 array of graphs the shape parameter of the EP distribution of X varies across rows with bx equal to -0.5 in the top row, then 0 and 0.5. The shape parameter of the EP distribution of measurement error, V, varies across columns with bv equal to -0.5 in the left column, then 0 and 0.5. Thus the centre pane on each page shows QRFs when both X and V are normally distributed.

First consider the *exact* error contaminated QRFs (solid lines). Attenuation (around 25%) is evident in every case. The exact error contaminated QRFs are non-

[12] An easily computed expression for the EP τ-qauntiles is given in Appendix 3.
[13] The R procedure *integrate* is used , R Core Team (2016).
[14] The R procedure *uniroot* is used, R Core Team (2016).

linear except when X and V are both normally distributed although the nonlinearity is very weak when the error free covariate is normal (centre rows).

Varying the shape of the distribution of W (compare graph arrays) and V (compare columns) has *little* effect on the error contaminated QRFs. Varying the shape of the distribution of the error free covariate X (compare rows) has a *substantial* effect. When this distribution is peaked (bottom rows) attenuation is most marked at values of Z near the centre of the distribution of X. When it is platykurtic (top rows) attenuation is most marked for values of z in the tail area of the distribution of X.

The shapes of the error contaminated QRFs vary little as τ is altered. The additional noise introduced by measurement error moves the QRFs away from the median QRF.

Now consider the *approximate* error contaminated QRFs (dashed lines). These are calculated using (9) with $g_Z^j(z)$ in place of $g_X^j(z)$ because it is in this form that the approximation is used in the sensitivity analysis described in Section 2.3. In every case the approximation accurately captures the attenuation and nonlinearity in the error contaminated QRF. The location of the error contaminated QRF is very accurately captured by the approximate median regressions ($\tau = 0.5$). The approximate QRFs for $\tau > 0.5$ tend to be located a little above the exact QRFs for $\tau > 0.5$ and below for $\tau < 0.5$. The quality of the approximations varies only a little as the three EP shape parameters are altered.

In summary, with linear error free QRFs, in the cases studied, error contaminated QRFs are significantly nonlinear unless the error free covariate is normally distributed. The main QRF deforming impact of measurement error is driven by the shape of the distribution of the error free covariate. When the variance of measurement error is not too large, this shape is reflected in the shape of the distribution of the error contaminated covariate which is the driving force in the approximation (9). As a result the approximation captures the nonlinearity in the error contaminated QRFs well, although there is some error in locating the vertical location of the extreme QRFs. In the "bias correction" procedure and sensitivity analysis developed in Section 5 this location error has little impact because data on Y is used to "calibrate" the locations of the QRFs.

5. Bias correction and sensitivity analysis

Small variance approximations like that developed here can be used to gauge the sensitivity of estimators to varying amounts of measurement error. Examples are provided in Chesher and Santos Silva (2002), Chesher and Schluter (2002) and Battistin and Chesher (2014). In this Section we examine the potential of small variance approximations in this regard in the context of QRF estimation.

Suppose a parametric form of a QRF is specified - here a simple case is considered in which error free QRFs are linear and parallel so that Y is generated by the location shift model (12), but the method is more generally applicable. An alternative approach suitable when there is a nonparametric specification is proposed later in this Section. The τ-QRF of Y given X considered now is

$$Q_X(\tau, x) = \beta_0 + \beta_1 x + \sigma_W Q_W(\tau)$$

where $Q_W(\tau)$ is the τ-quantile of W.

The results in Section 3.2 give the following expression for the approximate error contaminated τ-QRF.

$$\begin{aligned} \tilde{Q}_Z(\tau, z) &= \beta_0^* + \beta_1 \left(z + \sigma^2 g_Z^z(z)\right) \\ \beta_0^* &= \beta_0 + \sigma_W Q_W(\tau) - \frac{\sigma^2}{2\sigma_W}\beta_1^2 g_W^w(Q_W(\tau)) \end{aligned} \tag{15}$$

The function $g_Z^z(z)$ is the z-derivative of the log density of the error contaminated covariate, a function that can be estimated with the data available. It is used here taking up the suggestion in Section 2.3 where it is noted that substituting this function for $g_X^x(z)$ does not alter the order of the approximation error and allows realizations of Z to be used to estimate the approximate error contaminated QRF.

If the variance of measurement error were known then the error contaminated QRF could be estimated using $z + \sigma^2 \hat{g}_Z^z(z)$ as the right hand side variable regarding its estimated coefficient as an estimator of the slope of the error free QRF, β_1. If the approximation is accurate then we expect the inconsistency of this estimator to be small. The argument in Chesher and Santos Silva (2002) suggests that the difference between the pseudo-true value of this estimate and the error free QRF coefficient, β_1, will be of order $o(\sigma^2)$.

In the absence of knowledge of σ^2 a sensitivity analysis could be conducted, fixing σ^2 at a sequence of values in some plausible range, estimating the parameters of (15) at each chosen value of σ^2.

The method proposed involves two step estimation with a nonparametric plug-in estimator used at the first stage but that plug-in estimate is determined entirely by realizations of the error contaminated covariate. The principle of conditionality suggests that we should make inference conditional on covariate's realised values. Following that principle, the impact of variation in the plug in estimate on the sampling variance of the QRF estimator is carried into the conditional (on the realised values of Z) standard errors through the realised values of Z and $\hat{g}_Z^z(z)$.

In order to examine the performance of a procedure of this sort the results of some Monte Carlo experiments are now reported. The error free QRF is linear with $\beta_0 = 0$, $\beta_1 = 1$, $\sigma_W = 1$ and the distributions of W, X and V are exponential power distributions with mean zero and shape parameters $bw, bx, bv \in \{-0.5, 0, +0.5\}$, a total of 27 cases in all. The variances of W and V were set to one and the variance of X was set to 3. At these settings the R^2 in an error free mean regression is 0.75 and the attenuation of the error contaminated mean regression is 25%, that is the OLS estimator of β_1 using error contaminated Z has probability limit equal to 0.75. In each experiment a sample size of 400 was used and there were 2000 replications.

Results of two types are shown. In the first the exact function $g_Z^z(z)$ is used in constructing the new right hand side variables. In the second an estimate the function using an exponential series estimator is employed.

5.1. Log density derivative $g_Z^x(z)$ known. There are three tables of results, each showing means and standard deviations of estimates across the 2000 Monte Carlo replications[15]. The first, second and third sets of 9 rows show results for τ equal to respectively 0.5 (median regression), 0.75 and 0.90.

Table 1 shows results for the QRF estimator ignoring measurement error. The attenuation effect of measurement error is plain to see. In all cases the mean of the estimates of β_1 is very close to 0.75. The standard deviation of the estimates increases as τ increases as one would expect from the sampling theory of QRF estimators. There is little variation in the average value of the QRF estimator across values of the EP distribution shape parameters and across τ-QRFs.

Table 2 shows results for the QRF estimator with σ^2 "known". The improvement is substantial. The mean of the estimates of β_1 is very close to 1 (the error free QRF value), deviating at most by 3.6%. The accuracy of estimation is slightly impaired - the standard deviations of the measurement error "corrected" estimates are around 25% higher than the standard deviations of the naive estimator which ignores measurement error. There is a small amount of variation as the EP distribution shape parameters are altered. In the case in which the measurement error distribution is platykurtic the slope estimates are slightly downward biased at $\tau = 0.5$ and slightly upward biased at $\tau = 0.9$. There is the opposite effect when the measurement error distribution is leptokurtic with slight upward bias at $\tau = 0.5$ and slight downward bias at $\tau = 0.9$. These biases are, in all cases, very small.

Table 3 shows results when σ^2 is "estimated". When X is normally distributed there is extreme multicollinearity between z and $g_Z^z(z)$ and results are not shown for this case. When measurement error is also normally distributed $g_Z^z(z) \propto z$ and σ^2

[15] The sampling distributions seem close to symmetric, with means very close to medians, the latter thus not reported.

cannot be identified from the approximate QRF, or indeed at all.

Estimating σ^2 brings significant degradation in performance and now we find that one of the EP distribution shape parameters has a substantial influence, the shape parameter for the distribution of measurement error. The results vary only a little as the other shape parameters and τ are altered. With normal measurement error ($\gamma_V = 0$) the average of the "corrected" slope estimates is still very close to 1, deviating at most by 2.9%. With $\gamma_V = -0.5$, in which case the measurement error distribution is distinctly platykurtic, the "corrected estimates" are around 15% downward biased (compared with 25% for the naive estimator). With $\gamma_V = +0.5$ (leptokurtic) there is around 8% upward bias.

When σ^2 is estimated there is degradation in accuracy, standard deviations of the slope estimates increasing roughly fourfold. This is an effect that can be driven down by using larger samples. Of course in situations when $g_Z^z(z)$ is highly nonlinear this problem will be eased, but note that for real benefit to arise, this should be a nonlinearity arising from the distribution of error free X - if it arises from the distribution of V then the residual bias is likely to be large.

5.2. Log density derivative $g_Z^x(z)$ unknown. There are two sets of tables, laid out as described in the previous section. Table 4 gives results with σ^2 known and Table 5 gives results with σ^2 unknown.

The estimated log density derivative $g_Z^z(z) \equiv \frac{\partial}{\partial z} \log f_Z(z)$ is derived from the exponential series density estimator of Barron and Sheu (1993). The data are mapped by affine transformation onto the unit interval[16] and the unknown density of z is specified as

$$f_Z(z) \propto f_Z^0(z) \exp\left(\sum_{j=1}^{m} \theta_j h_j(z)\right) \tag{16}$$

where $f_Z^0(z) = 1$ is the uniform kernel density on $[0, 1]$ and the $h_j(\cdot)$ is the jth order

[16] The minimum and maximum of the realised values of Z are associated with respectively 0.1 and 0.9 to avoid end effects.

Legendre polynomial. The required log density derivative is simply

$$g_Z^z(z) = \sum_{j=1}^{m} \theta_j h_j'(z) \tag{17}$$

where $h_j'(\cdot)$ is the first derivative of the jth order Legendre polynomial.

The parameters θ are estimated by maximising a likelihood function in which (16) specifies the likelihood contributions up to a constant of integration found by numerical methods[17]. We choose $m = 8$ to produce the results given here. In a truly nonparametric estimation one would regard m as a smoothing parameter and determine a data driven appropriate value, for example by cross validation. In these Monte Carlo experiments m was fixed at a value which allowed the essential features of the density of Z to be captured while avoiding excessive roughness in the estimate.

First consider the case in which σ^2 is known and compare Tables 2 and 4. It is clear that estimating $g_Z^z(\cdot)$ has little effect on the bias of the measurement error corrected slope estimator, but it does slightly reduce the accuracy of the estimator, standard deviations across Monte Carlo replications rising by around 20%.

When σ^2 is estimated (compare Tables 3 and 5) the standard deviations of the slope estimates rise by two to four fold compared with the case when σ^2 is known and $g_Z^z(\cdot)$ is estimated, and by around 15% compared with the case in which σ^2 is estimated and $g_Z^z(\cdot)$ is known.

There is a significant increase in bias which is downward in all the cases considered. Since $\hat{g}_Z^z(z)$ is $g_Z^z(z)$ contaminated with measurement error, this could itself be a measurement error effect. Much smaller bias is found using smaller values of the smoothing parameter[18], m, but then the variance of the measurement error corrected estimator is much larger. If an attempt at estimating the measurement error variance is to be made, then, to avoid attenuation it seems to be important not to undersmooth

[17] Further details of the implementation of this procedure can be found in Chesher (1998b). The Monte Carlo experiments were conducted using R (R Core Team (2016)). In the density estimation, maximum likelihood estimation was done using the `nlm` procedure in R. QRFs were estimated using the procedure `rq` in the R contributed package `quantreg` (Koenker (2013)).

[18] For example the bias is reduced by around 50% on choosing $m = 4$.

when estimating $g^z_Z(z)$, and to have a large sample to hand.

5.3. Discussion. In the simple cases considered, estimation of approximate measurement error contaminated QRFs brings about a substantial reduction in bias but with an increase in variance that is small if the variance of measurement error is known, but sizeable otherwise. The proposed procedures are likely to work well in real problems only in large samples. But in many cases in microeconometric work in which QRF estimation would be contemplated large samples will be available, so perhaps this is not a great drawback.

Of more concern are the difficulties that would likely be encountered were more flexible forms of the error free QRF to be entertained. Once the error free QRF is specified as flexible and nonlinear there is the likelihood of collinearity between the derivatives of the error free QRF that appear in (6) and $g^z_Z(z)$. Another difficulty in nonlinear models is that if there are values of X at which the QRF is highly nonlinear then we can expect the approximation to have a large remainder term because it depends on the magnitude of the third derivatives of the error free QRF.

There is a further issue to consider. In practice QRFs are sometimes estimated in order to investigate heteroskedasticity. Dependence on X in the error free QRF that depends upon τ is manifested in the error contaminated QRF differently from dependence that is τ independent - see Section 3.3. To use the procedure developed here one must be specific about the interaction between X and τ in determining the error free QRF. In practice arriving at such a specification might be difficult and the resulting additional functions of z that arise may be highly collinear.

In the cases studied here there is a single covariate. Results in Chesher (1998b) for mean regression suggest that we can expect similarly good performance in multiple covariate problems as long as only one covariate is measured with error and the conditional density of the error contaminated covariates given the error free covariates depends on the latter through a single index.

An alternative procedure not investigated here, comes on using (5) to obtain an approximation for the error free QRF as follows

$$Q_X(\tau, z) = Q_Z(\tau, z) + \sum_{i,j} \sigma_{ij} \frac{F^i_{Y|Z}(Q_Z(\tau, z)|z) g^j_Z(z) + \frac{1}{2} F^{ij}_{Y|Z}(Q_Z(\tau, z)|z)}{F^Y_{Y|Z}(Q_Z(\tau, z)|z)} + o(\Sigma) \tag{18}$$

where the approximation

$$\sum_{i,j} \sigma_{ij} \left(\frac{F^i_{Y|X}(Q_Z(\tau, z)|z) g^j_X(z) + \frac{1}{2} F^{ij}_{Y|X}(Q_Z(\tau, z)|z)}{F^Y_{Y|X}(Q_Z(\tau, z)|z)} \right.$$
$$\left. - - \frac{F^i_{Y|Z}(Q_Z(\tau, z)|z) g^j_Z(z) + \frac{1}{2} F^{ij}_{Y|Z}(Q_Z(\tau, z)|z)}{F^Y_{Y|Z}(Q_Z(\tau, z)|z)} \right) = o(\Sigma)$$

proved as in Appendix 2, has been used. Taking this approach one investigates sensitivity to measurement error by calculating the right hand side of (18) using nonparametric estimators of quantile and density functions and of derivatives of distribution functions at a variety of conjectured values for Σ. This is similar to the method employed to produce measurement error corrected poverty indices in Chesher and Schluter (2002)

6. CONCLUDING REMARKS

Covariate measurement error causes fundamental changes in conditional quantile regression functions, altering their shape, orientation and location. This paper has provided information about the generic effects of measurement error by developing a small measurement error variance approximation to measurement error contaminated τ-QRFs. The approximation depends upon the error free QRF and its derivatives up to order two, the variance of measurement error, and the density of the error contaminated covariates. It does not depend upon, and to use it one needs no knowledge of, the specific form of the density of measurement error.

Exact calculations suggest that the approximation can be accurate when the amount of measurement error is small to moderate, as long as the error free QRF is not too nonlinear and the measurement error distribution is not too far from normal.

A number of uses of the approximation have been proposed.

1. It allows one to gauge the likely effects of measurement error on a particular form for an error free QRF that is proposed for use in analysis of data. With realizations of the error contaminated covariate one can estimate the terms in the approximation that depend on the density of this variate and, with a particular form for the error free QRF to hand, one can derive the remaining terms.
2. With knowledge of, or an estimate of, the variance of measurement error, it can be used to produce a measurement error corrected estimate of the parameters of the error free QRF.
3. It can be used to examine the sensitivity of QRF estimates to alternative assumed amounts of measurement error by estimating the approximate error contaminated QRF for a range of values of the measurement error variance.

APPENDIX 1: EXPRESSING APPROXIMATE QRFS AS FUNCTIONALS OF ERROR FREE QRFS

I use an abbreviated notation and consider conditional quantiles defined by the following equation

$$F(Q|x) = \tau \qquad \text{(A1.1)}$$

where Q denotes $Q(\tau, x)$ a dependence we make explicit in places where otherwise there might be confusion.

Considering variations in x, τ and Q subject to (A1.1) there is

$$F^Y(Q|x)dQ + \sum_i F^i(Q|x)dx_i = d\tau \qquad \text{(A1.2)}$$

where

$$\begin{aligned} F^Y(Q|x) &\equiv \frac{\partial}{\partial y} F(y|x)\bigg|_{y=Q} \\ F^i(Q|x) &\equiv \frac{\partial}{\partial x_i} F(y|x)\bigg|_{y=Q}. \end{aligned}$$

Shortly second partial derivatives appear, F^{YY}, F^{Yi} and F^{ij}, defined similarly. Equation (A1.2) leads directly to the following expressions for the first partial derivatives of the conditional quantile function.

$$Q^\tau(\tau, x) = \frac{1}{F^Y(Q|x)} \tag{A1.3}$$

$$Q^i(\tau, x) = -\frac{F^i(Q|x)}{F^Y(Q|x)} \tag{A1.4}$$

The second order partial derivatives of the quantile function follow on differentiating (A1.3) and (A1.4).

$$Q^{\tau\tau}(\tau, x) = -\frac{F^{YY}(Q|x)}{F^Y(Q|x)^2} Q^\tau(\tau, x) = -\frac{F^{YY}(Q|x)}{F^Y(Q|x)^3} \tag{A1.5}$$

$$\begin{aligned} Q^{\tau i}(\tau, x) &= -\frac{1}{F^Y(Q|x)^2}\left(F^{Yi}(Q|x) + F^{YY}(Q|x) Q^i(\tau, x)\right) \\ &= -\frac{F^{Yi}(Q|x)}{F^Y(Q|x)^2} + \frac{F^{YY}(Q|x)F^i(Q|x)}{F^Y(Q|x)^3} \end{aligned} \tag{A1.6}$$

$$\begin{aligned} Q^{ij}(\tau, x) &= -\frac{1}{F^Y(Q|x)}\left(F^{Yi}(Q|x) Q^j(\tau, x) + F^{ij}(Q|x)\right) \\ &\quad + \frac{F^i(Q|x)}{F^Y(Q|x)^2}\left(F^{YY}(Q|x) Q^j(\tau, x) + F^{Yj}(Q|x)\right) \\ &= -\frac{F^{ij}(Q|x)}{F^Y(Q|x)} + \frac{F^{Yi}(Q|x)F^j(Q|x)}{F^Y(Q|x)^2} + \frac{F^{Yj}(Q|x)F^i(Q|x)}{F^Y(Q|x)^2} \\ &\quad - \frac{F^{YY}(Q|x)F^i(Q|x)F^j(Q|x)}{F^Y(Q|x)^3} \end{aligned} \tag{A1.7}$$

In the main text we noted that

$$\left.\frac{\partial Q_Z}{\partial \sigma_{ij}}\right|_{\Sigma=0} = -\frac{F^i_{Y|X}(Q_Z|z)g^j_X(z)}{F^Y_{Y|X}(Q_Z|z)} - \frac{1}{2}\frac{F^{ij}_{Y|X}(Q_Z|z)}{F^Y_{Y|X}(Q_Z|z)} \tag{A1.8}$$

which we now wish to express in terms of the conditional QRF and its derivatives. The leading term is given directly by (A1.4) with suitable expansion of notation. Now note that, from (A1.6),

$$\frac{F^{Yi}(Q|x)F^j(Q|x)}{F^Y(Q|x)^2} = \frac{Q^{\tau i}(\tau,x)Q^j(\tau,x)}{Q^\tau(\tau,x)} - \frac{Q^{\tau\tau}(\tau,x)Q^i(\tau,x)Q^j(\tau,x)}{Q^\tau(\tau,x)^2}.$$

and from (A1.7), exploiting (A1.3) and (A1.4)

$$\frac{F^{ij}(Q|x)}{F^Y(Q|x)} = -Q^{ij}(\tau,x) + \frac{Q^{\tau i}(\tau,x)Q^j(\tau,x)}{Q^\tau(\tau,x)} + \frac{Q^{\tau j}(\tau,x)Q^i(\tau,x)}{Q^\tau(\tau,x)} - \frac{Q^{\tau\tau}(\tau,x)Q^i(\tau,x)Q^j(\tau,x)}{Q^\tau(\tau,x)^2}$$

Substituting this final expression in (A1.8) gives equation (6) in the main text.

APPENDIX 2: THE EFFECT ON THE APPROXIMATION OF USING THE LOG DENSITY OF Z RATHER THAN X.

Chesher (1991) shows that the densities of Z and X satisfy

$$f_Z(z) = f_X(z) + \sum_{s,t} \sigma_{st} f^{st}_X(z) + o(\Sigma).$$

The log densities therefore satisfy

$$g_Z(z) = g_X(z) + \sum_{s,t} \sigma_{st} \frac{f^{st}_X(z)}{f_X(z)} + o(\Sigma)$$

and their derivatives satisfy

$$g_Z^j(z) = g_X^j(z) + \sum_{s,t} \sigma_{st} \left(\frac{f_X^{stj}(z)}{f_X(z)} - \frac{f_X^{st}(z) f_X^j(z)}{f_X(z)^2} \right) + o(\Sigma).$$

It follows immediately that

$$\sum_{i,j} \sigma_{ij} Q_X^i(\tau, z) g_X^j(z) - \sum_{i,j} \sigma_{ij} Q_X^i(\tau, z) g_Z^j(z) = o(\Sigma)$$

and then directly that the order of the approximation error in (6) is not increased on substituting $g_Z^j(z)$ for $g_X^j(z)$.

APPENDIX 3: EXPONENTIAL POWER DISTRIBUTIONS: QUANTILES AND RANDOM NUMBER GENERATION

Let S have an exponential power distribution with mean μ and variance λ^2 and shape parameter $b \in (-1, 1)$. The probability density function of S is as follows.

$$f_S(s) = A \exp\left(-B \left| \frac{s - \mu}{\lambda} \right|^{\frac{2}{1+b}} \right)$$

$$A = \tfrac{1}{\lambda} \left(\frac{\Gamma\left(\frac{3}{2}(1+b)\right)}{(1+b)\Gamma\left(\frac{1}{2}(1+b)\right)^{3/2}} \right)^{1/2} \qquad B = \left(\frac{\Gamma\left(\frac{3}{2}(1+b)\right)}{\Gamma\left(\frac{1}{2}(1+b)\right)} \right)^{\frac{1}{1+b}}$$

Let G have a Gamma distribution with mean and variance δ. The density function of G is

$$f_G(g) = \Gamma(\delta)^{-1} g^{\delta-1} \exp(-g), \qquad g \in [0, \infty]$$

Quantiles. Fast routines for calculating Gamma quantiles are easy to find. They can be used to calculate EP quantiles, as follows.

Let $Q_G(\tau; \delta)$ be the τ-quantile of G. Let $Q_S(\tau; \mu, \lambda, b)$ be the τ-quantile of exponential power distributed S. Quantiles of S are related to quantiles of G as follows.

$$Q_S(\tau;\mu,\lambda,b) = \mu + \lambda \operatorname{sign}(\tau - 0.5)\left(B^{-1}Q_G(1 - \frac{2\min(\tau, 1-\tau)\lambda^{1/2}}{(1+b)^{\frac{1}{2}}\Gamma\left(\frac{1}{2}(1+b)\right)^{\frac{3}{4}}}, \frac{1}{2}(1+b))\right)^{\frac{1+b}{2}}$$

Pseudo-random number generation. The EP quantile formula leads directly to fast pseudo-random number generation because, if K has a uniform distribution on $[0,1]$, then $Q_S(K;\mu,\lambda,b)$ has an EP distribution with mean μ, variance λ^2 and shape parameter b.

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Table 1: Means and standard deviations of QRF slope estimates ignoring measurement error

			$bv = -0.5$		$bv = 0.0$		$bv = +0.5$	
τ	bw	bx	mean	s.d.	mean	s.d.	mean	s.d.
0.50	-0.5	-0.5	.738	.029	.755	.031	.772	.033
		0.0	.734	.031	.750	.033	.769	.034
		+0.5	.728	.034	.744	.035	.761	.038
	0.0	-0.5	.736	.030	.755	.031	.774	.032
		0.0	.732	.031	0.750	.033	.771	.034
		+0.5	.725	.034	.743	.035	.763	.035
	+0.5	-0.5	.736	.028	.756	.029	.778	.032
		0.0	.730	.030	.750	.032	.772	.033
		+0.5	.723	.033	.743	.034	.764	.037
0.75	-0.5	-0.5	.746	.034	.753	.034	.764	.036
		0.0	.742	.034	.750	.036	.761	.037
		+0.5	.739	.038	.747	.038	.757	.040
	0.0	-0.5	.746	.033	.752	.034	.763	.036
		0.0	.743	.034	.750	.036	.761	.037
		+0.5	.740	.036	.745	.038	.756	.039
	+0.5	-0.5	.747	.032	.753	.034	.763	.036
		0.0	.743	.034	.750	.035	.760	.037
		+0.5	.739	.036	.746	.038	.756	.039
0.90	-0.5	-0.5	.765	.042	.748	.044	.736	.044
		0.0	.766	.043	.750	.044	.740	.046
		+0.5	.769	.045	.754	.047	.743	.048
	0.0	-0.5	.766	.043	.747	.043	.735	.047
		0.0	.768	.044	.750	.044	.738	.047
		+0.5	.770	.045	.752	.045	.744	.048
	+0.5	-0.5	.770	.045	.746	.044	.733	.046
		0.0	.771	.044	.750	.045	.737	.047
		+0.5	.773	.046	.754	.046	.742	.048

Table 2: Means and standard deviations of measurement error corrected QRF slope estimates with σ^2 known and $g_Z^x(\cdot)$ known

			$bv = -0.5$		$bv = 0.0$		$bv = +0.5$	
τ	bw	bx	mean	s.d.	mean	s.d.	mean	s.d.
0.50	-0.5	-0.5	0.989	.040	1.011	.040	1.028	.042
		0.0	0.978	.042	1.000	.044	1.026	.046
		+0.5	0.972	.043	0.996	.046	1.021	.050
	0.0	-0.5	0.986	.041	1.010	.040	1.031	.041
		0.0	0.976	.041	1.000	.043	1.028	.045
		+0.5	0.970	.044	0.995	.046	1.024	.047
	+0.5	-0.5	0.987	.039	1.013	.038	1.036	.041
		0.0	0.974	.040	1.000	.043	1.030	.044
		+0.5	0.966	.042	0.995	.044	1.025	.048
0.75	-0.5	-0.5	0.994	.045	1.007	.044	1.018	.046
		0.0	0.989	.046	1.000	.047	1.015	.050
		+0.5	0.988	.049	0.998	.050	1.011	.053
	0.0	-0.5	0.992	.044	1.005	.044	1.018	.046
		0.0	0.990	.046	1.000	.048	1.014	.049
		+0.5	0.988	.047	0.996	.049	1.013	.052
	+0.5	-0.5	0.993	.044	1.005	.044	1.018	.046
		0.0	0.991	.046	1.000	.047	1.014	.049
		+0.5	0.989	.047	0.997	.049	1.012	.052
0.90	-0.5	-0.5	1.004	.056	0.994	.058	0.984	.058
		0.0	1.020	.058	1.000	.059	0.986	.062
		+0.5	1.029	.058	1.005	.062	0.984	.064
	0.0	-0.5	1.005	.056	0.990	.057	0.982	.059
		0.0	1.023	.059	1.000	.059	0.984	.062
		+0.5	1.032	.059	1.003	.059	0.986	.063
	+0.5	-0.5	1.007	.059	0.988	.059	0.978	.059
		0.0	1.026	.059	1.001	.059	0.981	.062
		+0.5	1.036	.059	1.004	.059	0.984	.065

Table 3: Means and standard deviations of measurement error corrected QRF slope estimates with σ^2 unknown and $g_Z^x(\cdot)$ known

			$bv = -0.5$		$bv = 0.0$		$bv = +0.5$	
τ	bw	bx	mean	s.d.	mean	s.d.	mean	s.d.
0.50	-0.5	-0.5	0.870	0.107	1.024	.127	1.087	.130
		0.0	-	-	-	-	-	-
		+0.5	1.117	.168	1.017	.161	0.910	.149
	0.0	-0.5	0.867	.106	1.023	.122	1.095	.129
		0.0	-	-	-	-	-	-
		+0.5	1.123	.160	1.018	.161	0.909	.152
	+0.5	-0.5	0.874	.105	1.029	.120	1.101	.128
		0.0	-	-	-	-	-	-
		+0.5	1.122	.164	1.020	.158	0.908	.152
0.75	-0.5	-0.5	0.892	.121	1.013	.137	1.074	.142
		0.0	-	-	-	-	-	–
		+0.5	1.106	.180	1.008	.180	0.899	.161
	0.0	-0.5	0.888	.119	1.017	.133	1.078	.146
		0.0	-	-	-	-	-	-
		+0.5	1.098	.170	1.004	.175	0.903	.161
	+0.5	-0.5	0.890	.116	1.013	.136	1.073	.144
		0.0	-	-	-	-	-	-
		+0.5	1.102	.178	1.011	.170	0.903	.162
0.90	-0.5	-0.5	0.933	.152	0.988	.181	1.015	.188
		0.0	-	-	-	-	-	-
		+0.5	1.077	.218	0.988	.216	0.880	.194
	0.0	-0.5	0.931	.158	0.993	.169	1.020	.192
		0.0	-	-	-	-	-	-
		+0.5	1.066	.227	0.980	.221	0.886	.194
	+0.5	-0.5	0.934	.158	0.981	.182	1.013	.196
		0.0	-	-	-	-	-	-
		+0.5	1.064	.227	0.987	.217	0.887	.201

Table 4: Means and standard deviations of measurement error corrected QRF slope estimates with σ^2 known and $g_Z^x(\cdot)$ estimated

			$bv = -0.5$		$bv = 0.0$		$bv = +0.5$	
τ	bw	bx	mean	s.d.	mean	s.d.	mean	s.d.
0.50	-0.5	-0.5	0.979	.048	1.002	.049	1.024	.052
		0.0	0.972	.047	0.994	.050	1.021	.052
		+0.5	0.968	.047	0.991	.051	1.017	.056
	0.0	-0.5	0.977	.049	1.003	.049	1.027	.051
		0.0	0.969	.046	0.994	.049	1.024	.052
		+0.5	0.965	.048	0.991	.051	1.020	.052
	+0.5	-0.5	0.978	.048	1.005	.047	1.032	.051
		0.0	0.968	.047	0.993	.049	1.024	.052
		+0.5	0.963	.046	0.992	.049	1.021	.055
0.75	-0.5	-0.5	0.986	.053	0.999	.053	1.015	.055
		0.0	0.984	.051	0.993	.053	1.012	.056
		+0.5	0.986	.052	0.994	.054	1.008	.060
	0.0	-0.5	0.984	.052	0.999	.052	1.016	.055
		0.0	0.985	.051	0.993	.053	1.011	.057
		+0.5	0.986	.051	0.994	.053	1.009	.057
	+0.5	-0.5	0.986	.052	0.997	.052	1.015	.054
		0.0	0.986	.051	0.994	.054	1.010	.057
		+0.5	0.985	.051	0.994	.053	1.008	.057
0.90	-0.5	-0.5	0.999	.063	0.987	.064	0.979	.067
		0.0	1.015	.064	0.994	.064	0.983	.067
		+0.5	1.027	.063	1.003	.065	0.983	.068
	0.0	-0.5	0.999	.064	0.985	.064	0.977	.068
		0.0	1.019	.063	0.992	.064	0.980	.068
		+0.5	1.029	.061	1.002	.064	0.984	.067
	+0.5	-0.5	1.003	.064	0.983	.063	0.975	.067
		0.0	1.021	.064	0.997	.066	0.977	.069
		+0.5	1.032	.063	1.002	.063	0.982	.069

Table 5: Means and standard deviations of measurement error corrected QRF slope estimates with σ^2 unknown and $g_Z^x(\cdot)$ estimated

			$bv = -0.5$		$bv = 0.0$		$bv = +0.5$	
τ	bw	bx	mean	s.d.	mean	s.d.	mean	s.d.
0.50	-0.5	-0.5	0.820	.102	0.903	.136	0.972	.169
		0.0	-	-	-	-	-	-
		+0.5	0.944	.182	0.907	.170	0.863	.148
	0.0	-0.5	0.818	.101	0.906	.137	0.974	.173
		0.0	-	-	-	-	-	-
		+0.5	0.947	.181	0.904	.153	0.865	.156
	+0.5	-0.5	0.817	.097	0.908	.128	0.976	.188
		0.0	-	-	-	-	-	-
		+0.5	0.950	.172	0.906	.150	0.862	.147
0.75	-0.5	-0.5	0.835	.118	0.900	.152	0.958	.183
		0.0	-	-	-	-	-	–
		+0.5	0.940	.187	0.902	.180	0.845	.162
	0.0	-0.5	0.830	.116	0.903	.151	0.955	.187
		0.0	-	-	-	-	-	-
		+0.5	0.939	.187	0.888	.175	0.853	.180
	+0.5	-0.5	0.830	.117	0.896	.136	0.949	.196
		0.0	-	-	-	-	-	-
		+0.5	0.941	.178	0.896	.165	0.845	.168
0.90	-0.5	-0.5	0.856	.163	0.884	.173	0.906	.220
		0.0	-	-	-	-	-	-
		+0.5	0.939	.214	0.888	.212	0.824	.199
	0.0	-0.5	0.859	.158	0.883	.193	0.902	.222
		0.0	-	-	-	-	-	-
		+0.5	0.933	.218	0.878	.219	0.829	.214
	+0.5	-0.5	0.857	.155	0.878	.173	0.898	.235
		0.0	-	-	-	-	-	-
		+0.5	0.933	.214	0.883	.203	0.823	.206

Figure 1: Exact and approximate τ-QRFs: $\tau \in \{0.5, 0.75, 0.9\}$, $bw = +0.5$

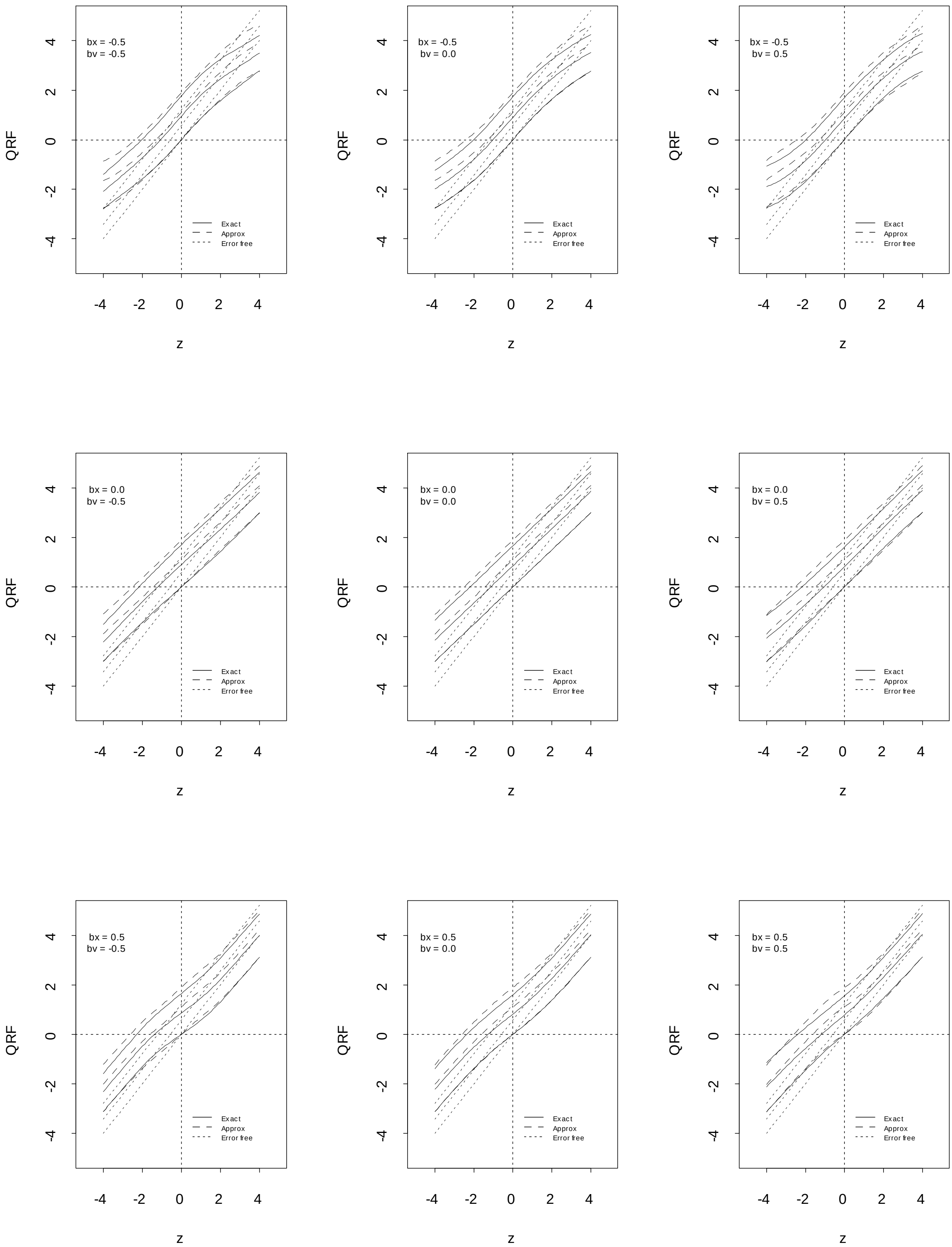

Figure 2: Exact and approximate τ-QRFs: $\tau \in \{0.5, 0.75, 0.9\}$, $bw = 0.0$

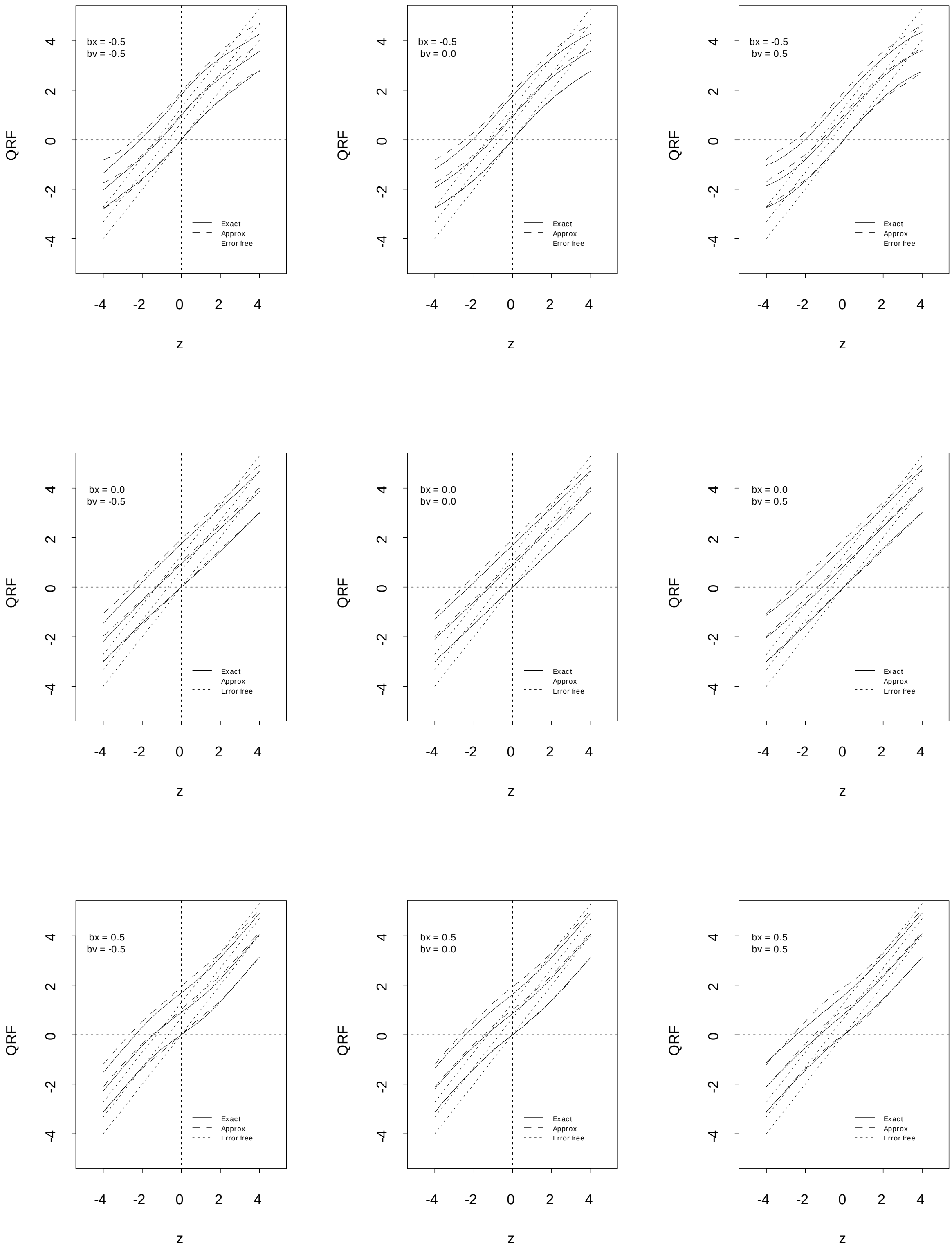

Figure 3: Exact and approximate QRFs: $\tau \in \{0.5, 0.75, 0.9\}$, $bw = -0.5$