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Optimal random taxation and redistribution

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Abstract

We assess the usefulness of stochastic redistribution among a continuum of risk-averse agents with quasilinear utilities in labor. Agents differ according to their consumption tastes, which remain private information. We identify circumstances where stochastic redistribution is socially dominated by the deterministic policy where after-tax income lotteries are replaced with their certainty equivalent. We also provide a parametric example where feasible and incentive compatible lotteries locally dominate the optimal deterministic menu. In this example the downward pattern of incentives prevailing in the deterministic case is reversed to an upward pattern in the stochastic case.

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1 Introduction

Solutions to optimization programs that we encounter in economic problems may be non deterministic because of failures of the standard convexity assumptions. Such failures often are due to the incentive constraints that appear in the presence of informational asymmetries. The allocation designed for a given agent then appears in the two sides of inequality constraints as it influences both her actual utility and the utility that would be obtained by those who stand ready to mimic this agent. Randomizing over allocations located on the frontier of the set delimited by the constraints, if allowed, leads to allocations on the convex hull of the constraints of the deterministic program. Non-convexities imply that the two sets, deterministic or random, differ and so open the possibility that some allocations in the convex hull socially improve upon the best deterministic alternative.[1]

Early studies in public finance in Weiss (1976), Stiglitz (1981), Stiglitz (1982) or Brito, Hamilton, Slutsky, and Stiglitz (1995) have exploited these ideas to show that deterministic redistribution sometimes is socially dominated. In the recent contributions of Lang (2017), Ederer, Holden, and Meyer (2018), Lang (2021) or Hines and Keen (2021), tax randomization makes incentive schemes opaque, which is used to limit gaming social rules; courts may for instance opt for vague standards and legal uncertainty to discourage some firms to undertake specific actions, or health authorities may prefer that hospitals are not fully aware of the exact reimbursement scheme to avoid that they reject sick patients and select instead healthy patients to get a higher reimbursement.[2]

The main argument developed in this literature is made especially insightful in Hellwig (2007) and Miller, Wagner, and Zeckauser (2010). Suppose that the government would like to redistribute income from high to low skill in a population of risk-averse workers. Redistribution is potentially limited if individual skills remain private information to workers, as high skilled may be willing to mimic low skilled by reducing labor effort. Introducing randomness in the income designed for low skilled is detrimental to their welfare, but this also relaxes incentive constraints and thus expands the scope of possible redistribution.[3]

[1]Such failures may also be due to the convexity of individual objectives, rather than non-convexity of the set of constraints. For instance, Hines and Keen (2021) use the convexity of profit with respect to price of a firm under perfect competition to show that tax uncertainty may be optimal.

[2]See also Pavlov (2011), Gauthier and Laroque (2014), Pycia and Unver (2015) or Gauthier and Laroque (2017) for a more mathematical approach.

[3]Similar arguments are also developed in industrial organization, following Gjesdal (1982). The revenue maximizing rule of a monopolist selling multiple goods to a single buyer may involve lotteries when the amount that buyers are willing to pay is privately known; see, e.g., Thanassoulis (2004), Manelli and Vincent (2010), Hart and Reny (2015) and Rochet and Thanassoulis (2019). In the context of optimal insurance design, Jeleva and Villeneuve (2004) derives conditions for insurers to offer random insurance contracts where reimbursements in a given state are made conditional to the realization of some other random circumstances. In a different vein, Bianchi and Jehiel (2015) shows that a principal may gain by

There is no reason why the direct welfare cost bearing on those who actually face noise would always overcome the gain from the expanded scope of redistribution. We suspect that this is more likely to happen if low skilled are strongly risk averse while high skilled do not suffer much from the noise, since then noise may not deter much the high skilled from mimicking the low skilled workers. Hellwig (2007) indeed shows that a pattern of risk aversion decreasing with labor productivity makes random redistribution socially useless in a Mirrlees optimal taxation model with a continuum of skill types of workers. The property is established in a rich setup where general preferences of the taxpayers are on multiple consumption goods, but social preferences are restricted to fit an unweighted Utilitarian criterion.

The role played by social redistributive concerns is not entirely clear. On the one hand, a high social valuation put on low skilled reinforces the argument in Hellwig (2007) as it tends to magnify the valuation by the society of the suffering of these agents when they face randomness in taxes and transfers. But, on the other hand, it also affects the reference scheme of deterministic redistribution where risk attitudes are evaluated and to which random redistribution will be compared.

In this paper, we reexamine the issue of social usefulness of random redistribution in the Mirrlees model. We simplify the exposition compared to Hellwig (2007) by assuming that individual preferences are quasilinear in labor, as in the classic analysis in Weymark (1987) or Lollivier and Rochet (1983). This allows us to deal with a more general redistributive stance by considering a government referring to a weighted Utilitarian objective, including the Rawlsian limit case. In this model we obtain a necessary and sufficient condition for improving upon a redistribution where income is random by proposing instead the certainty equivalent of the random income. We find that random redistribution is socially useless in the unweighted Utilitarian case of Hellwig (2007). More generally, random redistribution appears dominated if the government puts a higher social weight on the agents with the highest risk premia, which yields a natural generalization of Hellwig (2007). Still, the positive correlation between social valuation and risk aversion appears not necessary for random redistribution to be socially useless. Indeed, the level of risk aversion matters on top of its correlation with social valuation: random redistribution is found useless in a population of agents who are highly risk averse, a result that echoes those in Ederer, Holden, and Meyer (2018).

The certainty equivalent allows us to shed light on the shape of the redistributive stance yielding a deterministic optimal taxation rule. However, it is obviously a quite specific alternative in the sense that other deterministic policies may possibly dominate random taxes and transfers. The issue is whether the optimal deterministic redistribution

keeping agents uninformed about some aspects of the environment, i.e., to offer a menu of contracts that does not remove all sources of uncertainty.

policy dominates any menu of income lotteries. We show that this is indeed the case if the deterministic policy involves no bunching, that is the incentive constraints do not constrain the government to offer the same pair of before and after-tax income to a pool of different types of agents. We exploit this property to build an explicit parametric example where the optimal deterministic policy is socially dominated by a random policy where the lotteries consist of small local randomizations around the deterministic optimum. To the best of our knowledge, this is the first example of useful randomization in the Mirrlees optimal taxation model with a continuum of types of workers.

The example reveals a new channel through which noise operates in the redistribution process. The gain from the relaxation of the incentive constraints is found to work through a reversal of the incentive pattern. If the government restricts to a deterministic policy, then the incentives go downward, with every type envying the neighboring lower types. This familiar pattern is reversed when noise is introduced: the incentives now proceed upward, with every type instead envying the neighboring higher types. Local randomizations then make incentives fully aligned with social preferences. Although the reversal gives rise to a social gain, the noise remains as a cost since incentives still require that the most socially deserving bear random transfers. The net impact is shown to be positive in a specific parametrization of the economy where type follows a generalized Weibull distribution.

The paper proceeds as follows. Our setup is described in Section 2 and Section 3 analyzes incentives in the presence of income lotteries and derives conditions for incentive compatibility within a menu of lotteries to be transmitted to their certainty equivalent incomes. Section 4 then provides a necessary condition for the deterministic menu of certainty equivalent to be socially better than the original menu of lotteries. Some economic interpretation relying on a marginal argument is given in Section 5. The role played by bunching in the deterministic optimum is presented in Section 6 and exploited in the parametric example in Section 7.

2 General framework

There is a continuum of agents in a unit size population. Every agent is indexed by her type θ, which is a real parameter taking values in $\Theta = \left[\theta^{\inf}, \theta^{\sup}\right]$. The type has cumulative distribution function $F : \Theta \to [0, 1]$ and a well-behaved associated positive probability density function $f : \Theta \to \mathbb{R}_{++}$. The preferences of a type θ agent are represented by the quasilinear utility function

$$u(c, \theta) - y \tag{1}$$

when she earns a before-tax income y and pays the tax $y - c$. The after-tax income c is also her consumption. The function u is increasing, strictly concave in c and differentiable in θ. It also satisfies the Spence-Mirrlees condition that the cross-derivative $u''_{c\theta}(c,\theta)$ keeps a constant sign for all (c,θ), which will be taken as negative.

The government offers a menu of contracts $(\tilde{c}(\theta), \tilde{y}(\theta))$ that consists of pairs of after and before-tax income lotteries. The lottery $\tilde{c}(\theta)$ designed for a type θ agent is such that its owner receives an after-tax income smaller than c with probability $H(c,\theta)$, $c \in [c^{\inf}, c^{\sup}]$. The cumulative distribution function $H(c,\theta)$ is differentiable in θ for all (c,θ). The expected utility of type θ facing the lottery $(\tilde{c}(\theta), \tilde{y}(\theta))$ is

$$\tilde{V}(\theta) = \mathbb{E}\left[u(\tilde{c}(\theta), \theta) - \tilde{y}(\theta)\right].$$

Feasibility requires

$$\int_{\Theta} \mathbb{E}\left[\tilde{c}(\theta) - \tilde{y}(\theta)\right] \mathrm{d}F(\theta) \leq 0. \tag{2}$$

If θ is private information to the agent, the government must ensure that every agent chooses the contract designed for her. This is satisfied if the incentive constraints

$$\mathbb{E}\left[u(\tilde{c}(\theta), \theta) - \tilde{y}(\theta)\right] \geq \mathbb{E}[u(\tilde{c}(\tau), \theta) - \tilde{y}(\tau)] \tag{3}$$

hold true for all (θ, τ) in $\Theta \times \Theta$.

An optimal redistribution policy is a menu of lotteries $(\tilde{c}(\theta), \tilde{y}(\theta))$ that maximizes the social welfare objective

$$\int_{\Theta} \tilde{V}(\theta)\, \mathrm{d}G(\theta) \tag{4}$$

subject to the feasibility constraint (2) and the incentive constraints (3). The social weights embodied in $G(\cdot)$ are normalized so that they sum up to 1 in the population.

3 Dealing with incentives

Let us first state conditions for incentive compatibility when agents face income lotteries, rather than the deterministic bundles that are usually considered in the main strand of the literature.

Lemma 1. *Consider a menu where the after-tax income lottery $\tilde{c}(\theta)$ is designed for a type θ agent. The incentive constraints (3) are satisfied only if*

$$\tilde{V}'(\theta) = \mathbb{E}\left[u'_{\theta}(\tilde{c}(\tau), \theta)\right] \tag{5}$$

and

$$\frac{\partial}{\partial \tau}\mathbb{E}\left[u'_{\theta}(\tilde{c}(\tau), \theta)\right] \geq 0 \tag{6}$$

for all θ and $\tau = \theta$. These conditions are sufficient for incentive compatibility if (6) holds true for all θ and τ.

Proof. The proof reproduces standard arguments used in the case of deterministic contracts; see, e.g., Section 2.3 in Salanié (2017). The incentive constraints (3) can be rewritten as

$$\theta = \arg\max_{\tau} \mathbb{E}[u(\tilde{c}(\tau), \theta) - \tilde{y}(\tau)]$$

for all θ. This requires that the truthful report $\tau = \theta$ is a local extremum of the utility $\mathbb{E}[u(\tilde{c}(\tau), \theta) - \tilde{y}(\tau)]$, i.e.,

$$\frac{\partial}{\partial \tau} \mathbb{E}[u(\tilde{c}(\tau), \theta) - \tilde{y}(\tau)] = 0 \tag{7}$$

at $\tau = \theta$. Using the envelope theorem, this is equivalent to (5).

In addition, truthful reporting $\tau = \theta$ must be a local maximizer of the utility. This is the case if $\mathbb{E}[u(\tilde{c}(\tau), \theta) - \tilde{y}(\tau)]$ is locally concave in τ at the extremum $\tau = \theta$. If (7) holds at $\tau = \theta$ for all θ, we have

$$\begin{aligned} \frac{\partial^2}{\partial \tau^2} \left(\mathbb{E}\left[u(\tilde{c}(\tau), \theta)\right] - \mathbb{E}\left[\tilde{y}(\tau)\right] \right) &= -\frac{\partial^2}{\partial \tau \partial \theta} \left(\mathbb{E}\left[u(\tilde{c}(\tau), \theta)\right] - \mathbb{E}\left[\tilde{y}(\tau)\right] \right) \\ &= -\frac{\partial}{\partial \tau} \mathbb{E}\left[u'_\theta(\tilde{c}(\tau), \theta)\right] \end{aligned}$$

at $\tau = \theta$. Local concavity thus reads as (6).

Conditions (5) and (6) are necessary and sufficient for $\mathbb{E}[u(\tilde{c}(\tau), \theta) - \tilde{y}(\tau)]$ to stand below $\mathbb{E}[u(\tilde{c}(\theta), \theta) - \tilde{y}(\theta)]$ for all τ close to θ. They do not ensure that truthful reporting is a global maximum for the utility. A sufficient condition for a global maximum obtains by observing that, using (7) with $\theta = \tau$,

$$\frac{\partial}{\partial \tau} \left(\mathbb{E}\left[u(\tilde{c}(\tau), \theta)\right] - \mathbb{E}\left[\tilde{y}(\tau)\right] \right) = \int_{\tau}^{\theta} \frac{\partial}{\partial \tau} \mathbb{E}\left[u'_\theta(\tilde{c}(\tau), z)\right] dz.$$

If (6) holds true for all τ and θ, then the right-hand side of this equality has the same sign as $\theta - \tau$, which implies that $\mathbb{E}\left[u(\tilde{c}(\tau), \theta) - \tilde{y}(\tau)\right]$ is single-peaked in τ with a global maximum attained at $\tau = \theta$. ■

In the case where the redistribution policy only involves deterministic contracts, every type θ is offered some bundle $(c(\theta), y(\theta))$ with certainty and gets utility

$$V(\theta) = u(c(\theta), \theta) - y(\theta).$$

Appealing to the Spence-Mirrlees condition, the second-order condition (6) for local incentive compatibility rewrites

$$\frac{\partial}{\partial \tau} u'_\theta(c(\tau), \theta) = u''_{c\theta}\left(c(\tau), \theta\right) c'(\tau) \geq 0 \Leftrightarrow c'(\tau) \leq 0, \tag{8}$$

for all τ. The incentive constraints corresponding to a deterministic menu where the after-tax income is non-increasing are thus satisfied if and only if (5) holds true, which is $V'(\theta) = u'_\theta(c(\theta), \theta)$ for all θ.

Consider a menu of lotteries $(\tilde{c}(\theta), \tilde{y}(\theta))$ satisfying incentive compatibility (3). We are interested into a characterization of menus of such lotteries where incentive compatibility is transmitted to menus of the associated certainty equivalent incomes. Recall that the certainty equivalent after-tax income $\mathbb{C}(\tilde{c}, \theta)$ of type θ when she faces the lottery $\tilde{c}$ is defined as the sure consumption such that

$$u\left(\mathbb{C}(\tilde{c}, \theta), \theta\right) = \mathbb{E}\left[u\left(\tilde{c}, \theta\right)\right].$$

The corresponding risk premium is $\pi(\tilde{c}, \theta) = \mathbb{E}\left[\tilde{c}\right] - \mathbb{C}(\tilde{c}, \theta)$.

Lemma 2. *Suppose that $\tilde{c}(\theta_1)$ first-order stochastically dominates $\tilde{c}(\theta_2)$ for any two types θ_1 and $\theta_2 > \theta_1$. Then inequality (6) holds true for all τ and θ. Furthermore, $\mathbb{C}(\tilde{c}(\theta_1), \theta) \geq \mathbb{C}(\tilde{c}(\theta_2), \theta)$ for all θ.*

Proof. We have

$$\frac{\partial}{\partial \tau}\mathbb{E}\left[u'_\theta\left(\tilde{c}(\tau), \theta\right)\right] = \int_{c^{\inf}}^{c^{\sup}} u'_\theta\left(c, \theta\right) \mathrm{d}H'_\theta(c, \tau).$$

An integration by parts yields

$$\int_{c^{\inf}}^{c^{\sup}} u'_\theta\left(c, \theta\right) \mathrm{d}H'_\theta(c, \tau) = \left[u'_\theta\left(c, \theta\right) H'_\theta(c, \tau)\right]_{c^{\inf}}^{c^{\sup}} - \int_{c^{\inf}}^{c^{\sup}} u''_{\theta c}\left(c, \theta\right) H'_\theta(c, \tau)\, \mathrm{d}c.$$

Since $H(c^{\inf}, \tau) = 0$ and $H(c^{\sup}, \tau) = 1$ for all τ, we have $H'_\theta(c^{\inf}, \tau) = H'_\theta(c^{\sup}, \tau) = 0$ for all τ. It follows that

$$\int_{c^{\inf}}^{c^{\sup}} u'_\theta\left(c, \theta\right) \mathrm{d}H'_\theta(c, \tau) = -\int_{c^{\inf}}^{c^{\sup}} u''_{\theta c}\left(c, \theta\right) H'_\theta(c, \tau)\, \mathrm{d}c.$$

The Spence-Mirrlees condition $u''_{\theta c}\left(c, \theta\right) < 0$ for all (c, θ) implies that (6) holds true for all τ and θ if $H'_\theta(c, \theta) \geq 0$ for all (c, θ), i.e., $H(c, \theta_1) \leq H(c, \theta_2)$ for all c, θ_1 and $\theta_2 \geq \theta_1$. This corresponds to the case where $H(c, \theta_1)$ first-order stochastically dominates $H(c, \theta_2)$. Since $u(c, \theta)$ is increasing with c, the expected utility of any type θ is greater with the lottery $H(c, \theta_1)$ than with $H(c, \theta_2)$. This yields the ranking of certainty equivalent incomes announced in the lemma. ■

Lemma 2 provides us with a natural generalization of the condition (8) for incentive compatibility in a stochastic environment. Under the Spence-Mirrlees condition, the monotonicity of the deterministic consumption with type is replaced with a first-order stochastic dominance ranking of lotteries. That is, if the lotteries satisfy the stochastic dominance ranking in Lemma 2, then incentive compatibility obtains if and only if the first-order local conditions (5) are satisfied. The lemma thus gives conditions for the validity of the so-called 'first-order approach' to deal with incentives when agents face a menu of lotteries.

However, the implied ranking of the certainty equivalent incomes associated with these lotteries is not enough for a similar statement to be valid in the case of the deterministic menu $(\mathbb{C}(\tilde{c}(\theta),\theta))$. Indeed the lemma shows that any given agent θ will find the certainty equivalent associated with the dominant lottery $\tilde{c}(\theta_1)$ more attractive, but it brings no information on how different types would value this certainty equivalent income. Lemma 3 takes a first step on this issue by looking at preferences of different types of agents facing the same lottery.

Lemma 3. *If $u'''_{\theta cc}(c,\theta) \leq 0$ for all (c,θ), then $\mathbb{C}(\tilde{c},\theta_1) > \mathbb{C}(\tilde{c},\theta_2)$ for all $\tilde{c}$ and any two types θ_1 and $\theta_2 > \theta_1$.*

Proof. The risk premium $\pi(\tilde{c},\theta)$ is positive since $u(c,\theta)$ is strictly concave in c. It is such that $\mathbb{E}[u(\tilde{c},\theta)] = u(\mathbb{E}[\tilde{c}] - \pi(\tilde{c},\theta),\theta)$ for every θ. By differentiation, we have

$$\pi'_\theta(\tilde{c},\theta)u'_c\left(\mathbb{E}\left[\tilde{c}\right] - \pi(\tilde{c},\theta),\theta\right) = u'_\theta\left(\mathbb{E}\left[\tilde{c}\right] - \pi(\tilde{c},\theta),\theta\right) - \mathbb{E}\left[u'_\theta\left(\tilde{c},\theta\right)\right].$$

Note that the derivative $\pi'_\theta(\tilde{c},\theta)$ is well-defined since $u'_c(c,\theta)$ is non-zero. With $\pi(\tilde{c},\theta) > 0$, the Spence-Mirrlees condition $u''_{\theta c}(c,\theta) < 0$ for all (c,θ) gives

$$u'_\theta\left(\mathbb{E}\left[\tilde{c}\right] - \pi(\tilde{c},\theta),\theta\right) > u'_\theta\left(\mathbb{E}\left[\tilde{c}\right],\theta\right).$$

If $u'_\theta(c,\theta)$ is concave in c, i.e., $u'''_{\theta cc}(c,\theta) \leq 0$ for all (c,θ), then Jensen's inequality leads to

$$u'_\theta\left(\mathbb{E}\left[\tilde{c}\right],\theta\right) \geq \mathbb{E}\left[u'_\theta\left(\tilde{c},\theta\right)\right].$$

Therefore

$$u'_\theta\left(\mathbb{E}\left[\tilde{c}\right] - \pi(\tilde{c},\theta),\theta\right) - \mathbb{E}\left[u'_\theta\left(\tilde{c},\theta\right)\right] > 0.$$

The result follows from $u'_c(c,\theta) > 0$ for all (c,θ) and $\mathbb{C}'_\theta(\tilde{c},\theta) = -\pi'_\theta(\tilde{c},\theta)$. ■

The negative sign of the third cross-derivative introduced in Lemma 3 says that the second derivative of the utility with respect to consumption, which captures the curvature of the utility function, is decreasing with type. For risk-averse agents, this second derivative is negative: higher type agents therefore tend to display higher risk aversions. It follows that a given lottery yields more utility to low types (low risk aversion) than high types.

The combination of Lemma 2 and 3 shows that the sign of this derivative also matters for the implementation of a menu of deterministic contracts consisting of certainty equivalent incomes.

Corollary 1. *Suppose that $\tilde{c}(\theta_1)$ first-order stochastically dominates $\tilde{c}(\theta_2)$ for any two types θ_1 and $\theta_2 > \theta_2$. If $u'''_{\theta cc}(c,\theta) \leq 0$ for all (c,θ), then $\mathbb{C}(\tilde{c}(\theta),\theta)$ is non-increasing with θ.*

Proof. This directly follows from Lemma 2 and Lemma 3. ∎

Consider a reform that replaces a menu of lotteries $(\tilde{c}(\theta),\tilde{y}(\theta))$ with a menu where the certainty equivalent $\mathbb{C}(\tilde{c}(\theta),\theta)$ instead is given to type θ with probability 1. For the class of lotteries considered in Corollary 1, if $u''_{cc}(c,\theta)$ is decreasing with θ, then incentive compatibility at the outcome of the reform obtains if and only if the first-order conditions (5) hold true in the final (deterministic) situation, i.e., $V'(\theta) = u'_\theta(\mathbb{C}(\tilde{c}(\theta),\theta),\theta)$ for all θ.

4 Certainty equivalent domination

We are interested into conditions ensuring a deterministic optimal redistribution policy. This is an optimal redistribution policy where every contract $(\tilde{c}(\theta),\tilde{y}(\theta))$ yields the outcome $(c(\theta),y(\theta))$ with probability 1. Observe first that there is no loss in considering nonrandom before-tax income profiles $(y(\theta))$. Indeed, with quasilinear utility (1), replacing for every θ the lottery $\tilde{y}(\theta)$ with the sure outcome $\mathbb{E}[\tilde{y}(\theta)]$ affects neither the constraints (2) and (3) nor the objective (4).

Consider a switch from a feasible and incentive compatible menu of lotteries where type θ gets $\tilde{c}(\theta)$ to the deterministic menu that allocates to this type the after-tax income certainty equivalent $\mathbb{C}(\tilde{c}(\theta),\theta)$ with probability 1 while such an agent produces $y(\theta)-\delta(\theta)$ for some well-chosen differentiable profile $(\delta(\theta))$. The change in social welfare (4) equals

$$\int_{\theta^{\inf}}^{\theta^{\sup}} [u(\mathbb{C}(\tilde{c}(\theta),\theta),\theta) - y(\theta) + \delta(\theta))]\,\mathrm{d}G(\theta) - \int_{\theta^{\inf}}^{\theta^{\sup}} [\mathbb{E}\,[u(\tilde{c}(\theta),\theta)] - y(\theta)]\,\mathrm{d}G(\theta).$$

Using the definition of the certainty equivalent, this change reduces to

$$\int_{\theta^{\inf}}^{\theta^{\sup}} \delta(\theta)\ \mathrm{d}G(\theta). \tag{9}$$

Social welfare improves if agents with high social valuations enjoy a reduction in their before-tax income.

The final (deterministic) menu must fit (2) and (3). Let us first examine incentive compatibility requirements (3). By assumption the random menu $(\tilde{c}(\theta), y(\theta))$ satisfies the incentive constraints (3), i.e.,

$$\theta = \arg\max_{\tau} \mathbb{E}\left[u(\tilde{c}(\tau), \theta)\right] - y(\tau) = \arg\max_{\tau} u(\mathbb{C}(\tilde{c}(\tau), \theta), \theta) - y(\tau),$$

where the last equality follows from the definition of the certainty equivalent. By (5) in Lemma 1, truthful reporting $\tau = \theta$ for type θ agents requires

$$\mathbb{C}'_{\tau}(\tilde{c}(\tau), \theta) u'_c(\mathbb{C}(\tilde{c}(\tau), \theta), \theta) - y'(\tau) = 0 \tag{10}$$

when evaluated at $\tau = \theta$.

When faced with the deterministic income tax schedule, the incentive constraints (3) must be satisfied,

$$\theta = \arg\max_{\tau} u(\mathbb{C}(\tilde{c}(\tau), \tau), \theta) - y(\tau) + \delta(\tau)$$

for all θ and τ. This now requires

$$\left[\mathbb{C}'_{\tau}(\tilde{c}(\tau), \tau) + \mathbb{C}'_{\theta}(\tilde{c}(\tau), \tau)\right] u'_c(\mathbb{C}(\tilde{c}(\tau), \tau), \theta) - y'(\tau) + \delta'(\tau) = 0 \tag{11}$$

when evaluated at $\tau = \theta$. Using (10), this first-order condition simplifies to

$$\delta'(\theta) = -\mathbb{C}'_{\theta}(\tilde{c}(\theta), \theta) u'_c(\mathbb{C}(\tilde{c}(\theta), \theta), \theta). \tag{12}$$

By summation over types, the adjustment in before-tax income consistent with incentives can therefore be expressed as a function of certainty equivalent after-tax incomes,

$$\delta(\theta) = \delta(\theta^{\inf}) - \int_{\theta^{\inf}}^{\theta} \mathbb{C}'_{\theta}(\tilde{c}(z), z) u'_c(\mathbb{C}(\tilde{c}(z), z), z) \,\mathrm{d}z \tag{13}$$

for all θ.

The change in social welfare consistent with incentives obtains by reintroducing the expression of $\delta(\theta)$ found in (13) into (9). After using the integration by parts formula, the change in social welfare rewrites

$$\delta(\theta^{\inf}) - \int_{\theta^{\inf}}^{\theta^{\sup}} \frac{1 - G(\theta)}{f(\theta)} \mathbb{C}'_{\theta}(\tilde{c}(\theta), \theta) u'_c(\mathbb{C}(\tilde{c}(\theta), \theta), \theta) \,\mathrm{d}F(\theta). \tag{14}$$

For every type θ, the certainty equivalent incomes $\mathbb{C}(\tilde{c}(\theta), \theta)$ in (14) is fixed by the initial menu of lotteries $(\tilde{c}(\theta))$. Thus the only adjustment that is still unknown in (14) is the change in production $\delta(\theta^{\inf})$ required from the lowest type $\theta^{\inf}$. The value of $\delta(\theta^{\inf})$

obtains from the feasibility constraint (2) at equality. Replacing in (2) the sure after-tax income $\mathbb{C}(\tilde{c}(\theta), \theta)$ with the difference $\mathbb{E}[\tilde{c}(\theta)] - \pi(\tilde{c}(\theta), \theta)$, the feasibility constraint takes the form

$$\int_{\theta^{\inf}}^{\theta^{\sup}} [y(\theta) - \delta(\theta) - \mathbb{E}[\tilde{c}(\theta)] + \pi(\tilde{c}(\theta), \theta)] \, \mathrm{d}F(\theta) = 0.$$

Since the initial random menu $(\tilde{c}(\theta), y(\theta))$ is assumed to meet (2), this constraint becomes

$$\int_{\theta^{\inf}}^{\theta^{\sup}} [\pi(\tilde{c}(\theta), \theta) - \delta(\theta)] \, \mathrm{d}F(\theta) = 0.$$

Using the expression of $\delta(\theta)$ obtained in (13) and the identity $\mathbb{C}'_\theta(\tilde{c}(\theta), \theta) = -\pi'_\theta(\tilde{c}(\theta), \theta)$ gives

$$\delta(\theta^{\inf}) = \int_{\theta^{\inf}}^{\theta^{\sup}} \pi(\tilde{c}(\theta), \theta) \, \mathrm{d}F(\theta) - \int_{\theta^{\inf}}^{\theta^{\sup}} \frac{1 - F(\theta)}{f(\theta)} \pi'_\theta(\tilde{c}(\theta), \theta) u'_c(\mathbb{C}(\tilde{c}(\theta), \theta), \theta) \, \mathrm{d}F(\theta).$$

Reintroducing this expression of $\delta(\theta^{\inf})$ into the change in social welfare (14) yields the following result:

Proposition 1. *Consider a feasible and incentive compatible menu of lotteries $(\tilde{c}(\theta), y(\theta))$. Suppose that the change in social welfare (9) is positive, i.e.,*

$$\int_{\theta^{\inf}}^{\theta^{\sup}} \left[\pi(\tilde{c}(\theta), \theta) - \frac{G(\theta) - F(\theta)}{f(\theta)} \pi'_\theta(\tilde{c}(\theta), \theta) u'_c(\mathbb{C}(\tilde{c}(\theta), \theta), \theta)\right] \mathrm{d}F(\theta) > 0. \tag{15}$$

If $\mathbb{C}(\tilde{c}(\theta), \theta)$ is non-increasing in θ, then the deterministic menu where every type θ gets the sure outcome $\mathbb{C}(\tilde{c}(\theta), \theta)$ satisfies both feasibility and incentive compatibility, and it yields a higher social welfare than $(\tilde{c}(\theta), y(\theta))$.

Argument. Replacing the random menu with the deterministic one yields a social welfare improvement if and only if (9) is positive. Using feasibility (2) and necessary conditions for incentive compatibility (3) in the final (deterministic) situation, the change in social welfare is given by the left-hand side of (15). Under the Spence-Mirrlees condition, the monotonicity condition on $\mathbb{C}(\tilde{c}(\theta), \theta)$ ensures that the incentive constraints hold. If $\mathbb{C}(\tilde{c}(\theta), \theta)$ is increasing for some θ, then this type θ prefers reporting any type τ close different from θ, which implies a failure of incentive compatibility. Corollary 1 gives sufficient conditions for certainty equivalent after-tax incomes to decrease with θ. ■

Condition (15) fits the intuition that an economy consisting of agents who display high enough risk aversions tends to be immune from socially beneficial income tax randomizations. In this respect, this result may be seen as a partial converse of Ederer, Holden, and Meyer (2018) where the best redistribution policy involves income randomness whenever taxpayers have low risk aversions.

Still, Condition (15) shows that the level must be compared to the shape of risk aversions in the population. Indeed it is satisfied independently of the level of risk aversion if the risk premium is the same for every type, $\pi'_\theta(\tilde{c}, \theta) = 0$ for all $\tilde{c}$ and θ. Then income randomization is useless for screening and only enters the social objective as a cost bearing on risk averse types. This happens when all agents are identical, $u(c, \theta)$ does not depend on θ. But this is also consistent with some heterogeneity across agents, e.g., in the multiplicative utility formulation $u(c, \theta) = \theta v(c)$ used by Lollivier and Rochet (1983), where the certainty equivalent is defined by $v(\mathbb{C}(\tilde{c}, \theta)) = \mathbb{E}[v(\tilde{c})]$, and so both $\mathbb{C}(\tilde{c}, \theta)$ and $\pi(\tilde{c}, \theta)$ are independent of type.

Condition (15) also highlights that the exploitation of heterogeneity in risk aversion to introduce random perturbations into the tax system requires a strong enough redistribution motive. In the unweighted Utilitarian case where $G(\theta) = F(\theta)$ for every θ, every agent is weighted equally in the social welfare objective and Condition (15) is always satisfied. More generally, Condition (15) is satisfied if the product $[G(\theta) - F(\theta)]\, \pi'_\theta(\tilde{c}, \theta)$ is non-positive for all types, i.e., if the socially favored agents display higher risk aversions (captured by a higher risk premium). In this sense, Proposition 1 generalizes results in Hellwig (2007) in our special quasi-linear utility formulation (1).

CARA-Gaussian example. Type θ agents have CARA preferences

$$u(c, \theta) = -\frac{1}{\theta} \exp(-\theta c),$$

and they face Gaussian after-tax income lotteries $(\tilde{c}(\theta))$ with mean $m(\tilde{c}(\theta))$ and variance $v(\tilde{c}(\theta)) > 0$. Then Condition (15) is satisfied independently of the social tastes for redistribution embodied in $G(\theta)$. To show this property, recall that

$$\mathbb{E}[u(\tilde{c}(\tau), \theta)] = -\frac{1}{\theta} \exp\left[-\theta \left(m(\tilde{c}(\tau)) - \frac{\theta}{2} v(\tilde{c}(\tau))\right)\right].$$

This yields the certainty equivalent

$$\mathbb{C}(\tilde{c}(\tau), \theta) = m(\tilde{c}(\tau)) - \frac{\theta}{2} v(\tilde{c}(\tau))$$

and the risk premium

$$\pi(\tilde{c}(\tau), \theta) = \frac{\theta}{2} v(\tilde{c}(\tau)).$$

Hence inequality (15) is equivalent to

$$\int_{\theta^{\inf}}^{\theta^{\sup}} v(\theta)\left[\theta - (G(\theta) - F(\theta))\exp\left(-\theta\mathbb{C}(\tilde{c}(\theta),\theta)\right)\right] \mathrm{d}\theta > 0.$$

Since both $G(\theta) - F(\theta) \leq 1$ and $\exp\left(-\theta\mathbb{C}(\tilde{c}(\theta),\theta)\right) \leq 1$ for all θ (the after-tax income $\mathbb{C}(\tilde{c}(\theta),\theta)$ is non-negative), we have $(G(\theta) - F(\theta))\exp\left(-\theta\mathbb{C}(\tilde{c}(\theta),\theta)\right) \leq 1$ for all θ. For $\theta^{\inf} \geq 1$, every term in the sum is non-negative. ■

5 A marginal interpretation

The main trade-offs in Condition (15) can be better seen by considering a tax reform that removes after-tax income randomness for just a small part of the population. That is, the lottery $\tilde{c}(\theta)$ is replaced with the certainty equivalent $\mathbb{C}(\tilde{c}(\theta),\theta)$ for every type in some interval $[\underline{\theta},\overline{\theta}]$, $\overline{\theta} = \underline{\theta} + \mathrm{d}\theta$ for positive small $\mathrm{d}\theta$, while it remains unchanged for types outside this interval.

Incentives are affected for types within the interval concerned by the reform. But there is also a change in utility of types close below $\underline{\theta}$ when they contemplate mimicking type $\underline{\theta}$ since they now face the sure outcome $\mathbb{C}(\tilde{c}(\underline{\theta}),\underline{\theta})$ rather than the lottery $\tilde{c}(\underline{\theta})$. This calls for an adjustment in before-tax income close below $\underline{\theta}$, which in turn affects lower types in the distribution. A similar argument applies for types above $\overline{\theta}$ which spreads up to the top of the distribution.

A formal account for this argument proceeds as follows. Within the interval concerned by the reform, the before-tax income adjustment obeys (12),

$$\delta'(\theta) = -\mathbb{C}'_\theta(\tilde{c}(\theta),\theta)u'_c\left(\mathbb{C}(\tilde{c}(\theta),\theta),\theta\right).$$

Since, by assumption, the contracts offered to types outside this interval only change by the before-tax income adjustment, incentive compatibility (3) requires that $\delta(\theta)$ is set to some uniform amount $\underline{\delta}$ for every $\theta \leq \underline{\theta}$, and $\overline{\delta}$ for every $\theta \geq \overline{\theta}$.

One can link these two adjustments by relying on the first-order approximation $\bar{\delta} \simeq \underline{\delta} + \delta'(\underline{\theta})\,\mathrm{d}\theta$. Using (12), this is

$$\bar{\delta} \simeq \underline{\delta} - \mathbb{C}'_\theta(\tilde{c}(\underline{\theta}),\underline{\theta})u'_c\left(\mathbb{C}(\tilde{c}(\underline{\theta}),\underline{\theta}),\underline{\theta}\right)\,\mathrm{d}\theta. \quad (16)$$

The quantity $\underline{\delta}$ obtains from the feasibility constraint at the outcome of the reform: the total resources

$$\int_{\theta^{\inf}}^{\theta^{\sup}} [y(\theta) - \delta(\theta)]\,\mathrm{d}F(\theta)$$

must finance the total amount of redistributed consumption

$$\int_{\theta^{\inf}}^{\underline{\theta}} \mathbb{E}\left[\tilde{c}(\theta)\right] \, \mathrm{d}F(\theta) + \int_{\underline{\theta}}^{\underline{\theta}+\mathrm{d}\theta} \mathbb{C}(\tilde{c}(\theta), \theta) \, \mathrm{d}F(\theta) + \int_{\underline{\theta}+\mathrm{d}\theta}^{\theta^{\sup}} \mathbb{E}\left[\tilde{c}(\theta)\right] \, \mathrm{d}F(\theta).$$

A production above the aggregate redistributed consumption would waste resources: the best feasible reform is such that the total resources equals the total consumption to be financed. Replacing $\mathbb{C}(\tilde{c}(\theta), \theta)$ with $\mathbb{E}\left[\tilde{c}(\theta)\right] - \pi(\tilde{c}(\theta), \theta)$ and using (2) at equality, the feasibility constraint at the outcome of the reform rewrites

$$\int_{\theta^{\inf}}^{\theta^{\sup}} \delta(\theta) \, \mathrm{d}F(\theta) = \int_{\underline{\theta}}^{\underline{\theta}+\mathrm{d}\theta} \pi(\tilde{c}(\theta), \theta) \, \mathrm{d}F(\theta).$$

At the first-order for $\mathrm{d}\theta$ close to 0 we thus have

$$\underline{\delta} F(\underline{\theta}) + \bar{\delta}\,(1 - F(\underline{\theta})) \simeq \pi(\underline{\theta}) f(\underline{\theta}) \, \mathrm{d}\theta. \tag{17}$$

The system formed by (16) and (17) gives the two before-tax income adjustments $\underline{\delta}$ and $\bar{\delta}$ consistent with feasibility and incentive compatibility at the outcome of the reform. Substituting $\bar{\delta}$ from (16) into (17) yields

$$\underline{\delta} \simeq \left[\pi(\underline{\theta}) f(\underline{\theta}) + (1 - F(\underline{\theta}))\, \mathbb{C}'_\theta(\tilde{c}(\underline{\theta}), \underline{\theta}) u'_c\left(\mathbb{C}(\tilde{c}(\underline{\theta}), \underline{\theta}), \underline{\theta}\right)\right] \mathrm{d}\theta,$$

and then one can get $\bar{\delta}$ from (16).

From (9), the reform improves social welfare if $\underline{\delta} G(\underline{\theta}) + \bar{\delta}\,(1 - G(\underline{\theta})) > 0$. Using (16), this is

$$\underline{\delta} - (1 - G(\underline{\theta}))\, \mathbb{C}'_\theta(\tilde{c}(\underline{\theta}), \underline{\theta}) u'_c\left(\mathbb{C}(\tilde{c}(\underline{\theta}), \underline{\theta}), \underline{\theta}\right) \mathrm{d}\theta > 0, \tag{18}$$

or equivalently, using $\mathrm{d}\theta > 0$ and the expression of the before-tax income adjustment $\underline{\delta}$ derived above,

$$\pi(\underline{\theta}) f(\underline{\theta}) + (G(\underline{\theta}) - F(\underline{\theta}))\, \mathbb{C}'_\theta(\tilde{c}(\underline{\theta}), \underline{\theta}) u'_c\left(\mathbb{C}(\tilde{c}(\underline{\theta}), \underline{\theta}), \underline{\theta}\right) > 0.$$

We recognize the expression that is summed up in (15), with $\mathbb{C}'_\theta(\tilde{c}(\theta), \theta) = -\pi'_\theta(\tilde{c}(\theta), \theta)$.

The intuition is as follows. Since every agent gets under truthful report the certainty equivalent consumption of her initial lottery, the impact on social welfare goes through the reduction in production effort obtained from the relaxation of incentives. Suppose that the government puts a higher weight on low types. When a high type θ mimics a neighboring lower type τ, she gets the lottery $\tilde{c}(\tau)$ initially and $\mathbb{C}(\tilde{c}(\tau), \tau)$ at the outcome of the reform. But type θ was indifferent between the lottery $\tilde{c}(\tau)$ and the sure outcome

$\mathbb{C}(\tilde{c}(\tau),\theta)$. Therefore the switch to a certain environment relaxes the incentive constraint if $\mathbb{C}(\tilde{c}(\tau),\theta) \geq \mathbb{C}(\tilde{c}(\tau),\tau)$. That is, the lottery $\tilde{c}(\tau)$ yields more utility to type θ than type τ herself, which reflects a lower risk aversion of type θ. In this case the switch to a certain environment allows the government to extract more resources from type θ and/or reduce the labor effort of type τ while keeping incentive compatibility. This results in a social welfare improvement since the government gives little importance to the suffering of the high type θ.

6 Randomization and deterministic bunching

The reference of certainty equivalent income as an alternative to the lottery highlights the role played by the redistributive concern and the distribution of risk aversion in the population. However other deterministic menus may improve upon a menu of lotteries. We now reverse the perspective by studying whether the optimal deterministic redistribution policy can also be optimal within the larger class of menus of lotteries.

We first characterize an optimal 'relaxed' redistribution policy. Such a policy maximizes the social objective (4) subject to feasibility (2) and the first-order conditions (5) given in Lemma 1; it abstracts from (6). This relaxed policy accordingly provides us with an upper bound for attainable levels of social welfare in an optimal policy, where (3) replaces (5).

By summing over types the first-order conditions (5) for local incentive compatibility, the indirect utility of type θ rewrites

$$\tilde{V}(\theta) = \tilde{V}(\theta^{\inf}) + \int_{\theta^{\inf}}^{\theta} \mathbb{E}\left[u'_\theta(\tilde{c}(z), z)\right] \mathrm{d}z. \tag{19}$$

In turn the utility $\tilde{V}(\theta^{\inf})$ can be expressed as a function of the profile of lotteries $(\tilde{c}(\theta))$ by reintroducing the expression of $\tilde{V}(\theta)$ in (19) into the feasibility constraint (2). Using the integration by parts formula then gives

$$\tilde{V}(\theta^{\inf}) = \int_{\theta^{\inf}}^{\theta^{\sup}} \left(\mathbb{E}\left[u(\tilde{c}(\theta),\theta) - \tilde{c}(\theta)\right] - \frac{1-F(\theta)}{f(\theta)}\mathbb{E}\left[u'_\theta(\tilde{c}(\theta),\theta)\right]\right) \mathrm{d}F(\theta).$$

An expression for the social welfare consistent with feasibility (2) and the necessary first-order condition (5) for local incentive compatibility finally obtains by reintroducing the indirect utility (19) into (4). With $\tilde{V}(\theta^{\inf})$ given above, and after a further integration by parts, the social welfare objective (4) rewrites

$$\int_{\theta^{\inf}}^{\theta^{\sup}} \mathbb{E}\left[W(\tilde{c}(\theta),\theta)\right] \mathrm{d}F(\theta) \tag{20}$$

where

$$W(c,\theta) = u(c,\theta) - c - \frac{G(\theta) - F(\theta)}{f(\theta)} u'_\theta(c,\theta)$$

represents the virtual contribution of type θ to social welfare when this type faces an after-tax income c with certainty.

Consider the special case where the menu offered to agents only consists of deterministic contracts. In an optimal deterministic redistribution policy, the non-random after-tax income menu $(c^*(\theta))$ that maximizes

$$\int_{\theta^{\inf}}^{\theta^{\sup}} W(c(\theta),\theta) \, \mathrm{d}F(\theta)$$

subject to the condition $c(\theta)$ is non-increasing with θ. It involves no bunching if the after-tax income maximizing $W(c,\theta)$ is non-increasing with θ. In this case,

$$c^*(\theta) = \arg\max_c W(c,\theta) \tag{21}$$

for every type θ.

Proposition 2. *A random redistribution policy is socially useless if the optimal deterministic redistribution policy involves no bunching.*

Proof. We first show that social welfare is greater when evaluated in an optimal deterministic policy than in an optimal 'relaxed' policy. It follows from (21) that $W(c^*(\theta),\theta) \geq W(c,\theta)$ for all (c,θ). Thus $W(c^*(\theta),\theta) \geq \mathbb{E}\left[W(\tilde{c}(\theta),\theta)\right]$ for all $(\tilde{c}(\theta),\theta)$, and so there is no menu of lotteries improving upon the social welfare at the deterministic optimum,

$$\int_{\theta^{\inf}}^{\theta^{\sup}} \mathbb{E}\left[W(\tilde{c}(\theta),\theta)\right] \mathrm{d}F(\theta) \leq \int_{\theta^{\inf}}^{\theta^{\sup}} W(c^*(\theta),\theta) \, \mathrm{d}F(\theta).$$

for every menu $(\tilde{c}(\theta))$. To conclude the proof, we observe that the level of social welfare in an optimal redistribution policy cannot be greater than its level in an optimal 'relaxed' policy. ■

Proposition 2 says that a socially useful randomness in the redistribution policy cannot obtain if the deterministic optimum does not bunch different types of agents by treating them equally for incentive reasons.[4]

[4] The property is actually used as an intermediate step in the proof used by Hellwig (2007); see his Lemma 5.6.

7 An example of useful randomization

We exploit the property in Proposition 2 to build an explicit parametric example where random redistribution is socially useful. The utility of type θ, $\theta \in \left[\theta^{\inf}, \theta^{\sup}\right]$, $\theta^{\inf} = 0$, is

$$u(c, \theta) - y = \ln(c + \theta) - y. \tag{22}$$

The Spence-Mirrlees condition is satisfied, $u''_{c\theta}(c, \theta) < 0$ for all c and θ.

7.1 Deterministic optimum

Our statu quo is the optimal deterministic redistribution policy chosen by a Rawlsian government when θ remains private information to the agent. The indirect utility of a type θ agent is $V(\theta) = \ln(c(\theta) + \theta) - y(\theta)$ when she faces the pair $(c(\theta), y(\theta))$ with certainty. The incentive constraints can be written as

$$V(\theta) \geq V(\tau) + \ln(c(\tau) + \theta) - \ln(c(\tau) + \tau)$$

The informational rent $\ln(c(\tau) + \theta) - \ln(c(\tau) + \tau)$ is positive for $\tau < \theta$. That is, the government must leave a positive rent to the type θ agents in order to prevent them from mimicking lower type $\tau < \theta$ agents. This puts limits to the redistribution desired by the government. Indeed, the first-order condition (5) in Lemma 1 is

$$V'(\theta) = \frac{1}{c(\theta) + \theta} > 0$$

for all θ, so that the government is constrained to leave a higher rent to higher types while it would instead like to favor the least type.

Under Rawlsian social preferences, $G(\theta) = 1$ for all θ. It follows from (20) that the best deterministic menu of a Rawlsian government maximizes

$$V(\theta^{\inf}) = \int_{\theta^{\inf}}^{\theta^{\sup}} \left[\ln(c(\theta) + \theta) - c(\theta) - \frac{1 - F(\theta)}{f(\theta)} \frac{1}{c(\theta) + \theta}\right] \mathrm{d}F(\theta) \tag{23}$$

subject to the monotonicity condition

$$c'(\theta) \leq 0 \tag{24}$$

for all θ. In the sequel, we assume that the Mills ratio

$$m(\theta) = \frac{1 - F(\theta)}{f(\theta)}$$

satisfies

$$m'(\theta) - [1 + 4m(\theta)]^{1/2} > 0, \tag{25}$$

for all types. This requires an increasing Mills ratio, which is not satisfied for most common distributions. Still the inequality is met for all but a small negligible subset of high types if θ is distributed according to a variant of Weibull distribution.

Generalized Weibull distribution. Given two real positive shape parameters a and b, and a real positive scale parameter $s > 0$, θ is distributed according to a Generalized Weibull distribution if

$$1 - F(\theta) = \exp(1 - (1 + s\theta^b)^a)$$

for all $\theta > 0$. For $a < 1$ and $b \leq 1$ the associated hazard rate is monotone decreasing and therefore $m(\theta)$ is increasing for all $\theta > 0$. See Dimitrakopoulou, Adamidis, and Loukas (2007) for properties of this distribution. For $a = 0.9$, $b = 0.01$ and $s = 5$ the left-hand side of the inequality (25) is decreasing in θ, from positive to negative values. There is therefore a threshold $\bar{\theta}$ such that the inequality (25) is satisfied for all $\theta < \bar{\theta}$. With the specific values chosen for the parameters a, b and s, the threshold $\bar{\theta}$ is equal to 6.39. At the threshold $F(\bar{\theta}) = 98.4\%$, i.e., (25) is satisfied for all types of agents but a small negligible subset consisting of the highest types. ■

The inequality (25) implies that the after-tax income maximizing pointwise $V(\theta^{\text{inf}})$ in (23) is always increasing with θ, thus violating the monotonicity requirement (24). The government cannot give a higher after-tax income to low types. As detailed in the Appendix, its second-best alternative consists in bunching all types. The optimal deterministic redistribution policy involves full equalization of after-tax incomes: every type gets $c^*(\theta) = c^*$. The incentive constraints then require the same production from every agent, $y(\theta) = y^*$, and feasibility (2) reads $y^* = c^*$. The utility of the least type therefore equals $V(\theta^{\text{inf}}) = \ln(c^* + \theta^{\text{inf}}) - c^*$. It is maximized by setting $c^* = 1 - \theta^{\text{inf}} = 1$ (since $\theta^{\text{inf}} = 0$), which yields the value of the social welfare evaluated at the optimal deterministic redistribution policy $V(\theta^{\text{inf}}) = -1$.

7.2 Incentives and local tax randomization

We now examine whether there exist after-tax income lotteries $\tilde{c}(\theta) = c^* + \tilde{\varepsilon}(\theta)$ that improve upon the deterministic optimum. We consider local randomizations close to the deterministic optimum: every realization of $\tilde{\varepsilon}(\theta)$ is close to 0. The random perturbation $\tilde{\varepsilon}(\theta)$ has both mean and variance close to 0. These two moments are parametrized in such a way that $\mathbb{E}[\tilde{\varepsilon}(\theta)] = \lambda\mu(\theta)$ and $\text{var}[\tilde{\varepsilon}(\theta)] = \lambda v(\theta)$, with λ a positive real number close to 0, and $\mu(\theta)$ and $v(\theta) \geq 0$ arbitrary bounded real numbers. For these lotteries, the (second-order

Taylor expansion of the) expected utility of type θ when she chooses $\tilde{c}(\tau)$ writes

$$\mathbb{E}\left[u(c^* + \tilde{\varepsilon}(\tau), \theta)\right] \simeq u(c^*, \theta) + \lambda u'_c(c^*, \theta) \left(\mu(\tau) - \frac{A(c^*, \theta)}{2} v(\tau) \right), \tag{26}$$

where

$$A(c, \theta) = -\frac{u''_{cc}(c, \theta)}{u'_c(c, \theta)} = \frac{1}{c + \theta} > 0$$

is the coefficient of absolute risk aversion of type θ. Given the after-tax income c, lower types display a higher risk aversion, captured by a higher coefficient of risk aversion.

The analysis of incentive compatibility simplifies drastically if one restricts attention to the special case where the mean and the variance of the perturbation are linearly related. Actually we set

$$\mu(\theta) = v(\theta) \tag{27}$$

for all θ. For this class of lotteries, the sub-utility from consumption in (26) obtained by type θ when she chooses the contract designed for type τ becomes

$$\mathbb{E}\left[u(c^* + \tilde{\varepsilon}(\tau), \theta)\right] \simeq u(c^*, \theta) + \lambda u'_c(c^*, \theta) \left(1 - \frac{A(c^*, \theta)}{2}\right) v(\tau).$$

Remark 1. A menu of lotteries $(c^* + \tilde{\varepsilon}(\theta))$ where $\mathbb{C}(c^* + \tilde{\varepsilon}(\theta), \theta) = c^*$ for all θ does not meet (27). Indeed, from (26), $\mathbb{E}[u(c^* + \tilde{\varepsilon}(\theta), \theta)] = u(c^*, \theta)$ obtains if

$$\mu(\theta) \simeq \frac{A(c^*, \theta)}{2} v(\theta).$$

The certainty equivalent of lotteries that fit (27) therefore differs from the after-tax income c^* in the optimal deterministic menu. $\square$

The indirect utility $\tilde{V}(\theta) = \mathbb{E}\left[u(c^* + \tilde{\varepsilon}(\theta), \theta)\right] - y(\theta)$ of type θ when she faces the menu of randomized incomes can now be expressed as

$$\tilde{V}(\theta) \simeq u(c^*, \theta) + U(\theta), \tag{28}$$

where

$$U(\theta) = S(c^*, \theta) v(\theta) - y(\theta) \tag{29}$$

and

$$S(c, \theta) = \lambda u'_c(c, \theta) \left(1 - \frac{A(c, \theta)}{2}\right) = \frac{\lambda}{c + \theta} \left(1 - \frac{1}{2} \frac{1}{c + \theta}\right). \tag{30}$$

The incentive constraints (3) then become

$$\begin{aligned} U(\theta) = S(c^*, \theta) v(\theta) - y(\theta) &\geq S(c^*, \theta) v(\tau) - y(\tau) \\ &= U(\tau) + (S(c^*, \theta) - S(c^*, \tau)) v(\tau). \end{aligned} \tag{31}$$

for all τ and θ. Local income tax randomizations within the class (27) give rise to incentive constraints which have the familiar quasilinear textbook shape. By Lemma 1, with $U(\theta)$ defined in (29), local incentive compatibility is met if and only if, for all θ,

$$U'(\theta) = S'_\theta(c^*, \theta)v(\theta), \tag{32}$$

and

$$S'_\theta(c^*, \theta)v'(\theta) \geq 0.$$

Furthermore, if

$$\int_\tau^\theta S'_\theta(c^*, z)v'(\tau)\,\mathrm{d}z$$

has the same sign as $\theta - \tau$, then the conditions for local incentive compatibility are sufficient for (3) to hold for all admissible τ and θ.

Since, for any $\theta \geq 0$,

$$S'_\theta(c, \theta) = -\frac{\lambda}{(c+\theta)^2}\left(1 - \frac{1}{c+\theta}\right) \leq 0 \tag{33}$$

when evaluated at $c = c^* = 1$, we finally obtain:

Lemma 4. *Consider a menu where the government adds random perturbations $\tilde{\varepsilon}(\theta)$ to the optimal deterministic after-tax income c^* with mean $\lambda\mu(\theta)$ and variance $\lambda v(\theta)$ for some $\lambda \geq 0$ close to 0. Assume that $\mu(\theta) = v(\theta)$ for all θ. The incentive constraints (3) are satisfied if and only if the first-order conditions (32) are satisfied and $v'(\theta) \leq 0$ for all θ.*

Lemma 4 shows that incentive compatibility requires that the variance of the after-tax income is non-increasing with type. By (27), the mean after-tax income is also non-increasing. If the government designs tax lotteries, then the socially favored least type agents must receive a higher average transfer than in the deterministic case, but they have also to bear random taxes. Eventually, for λ close to 0 in (28), they must register the highest gain from the noise since $\tilde{V}'(\theta) = u'_\theta(c^*, \theta) + U'(\theta) \simeq u'_\theta(c^*, \theta) > 0$. Still it is not clear whether their utility can fall above $V(\theta^{\inf})$.

The incentive constraints (31) highlight a nice consequence of random taxes. The negative sign of $S'_\theta(c^*, \theta)$ implies a reversal compared to the situation where the government uses deterministic redistribution tools. The informational rent $(S(c^*, \theta) - S(c^*, \tau))v(\tau)$ is now positive for $\tau > \theta$, i.e., type θ now envies agents whose types stand above, and not below, its own type. In this example, random taxation allows the government to make incentives aligned with social preferences.

7.3 Socially useful tax randomization

One can now assess the social usefulness of random redistribution for a Rawlsian planner by computing the utility $\tilde{V}(\theta^{\inf})$ of the least type of agents in the presence of a small after-tax income noise and compare it with their utility $V(\theta^{\inf}) = -1$ evaluated at the optimal deterministic optimum.

We first express the before-tax income of type θ as a function of her indirect utility $\tilde{V}(\theta)$. From (28), we have

$$y(\theta) = u(c^*, \theta) + S(c^*, \theta)v(\theta) - \tilde{V}(\theta).$$

The expression of the indirect utility $\tilde{V}(\theta)$ obtains from the first-order necessary condition (32) for incentive compatibility when agents face local randomization of their after-tax income, $U'(\theta) = S'_\theta(c^*, \theta)v(\theta)$ for all θ, which yields

$$U(\theta) = U(\theta^{\inf}) + \int_{\theta^{\inf}}^{\theta} S'_\theta(c^*, z)v(z)\mathrm{d}z.$$

Hence, using (28),

$$\tilde{V}(\theta) = u(c^*, \theta) + U(\theta^{\inf}) + \int_{\theta^{\inf}}^{\theta} S'_\theta(c^*, z)v(z)\mathrm{d}z.$$

The feasibility constraint (2) reads

$$\int_{\theta^{\inf}}^{\theta^{\sup}} [c^* + \lambda\mu(\theta) - y(\theta)]\, \mathrm{d}F(\theta) = 0$$

$$\Leftrightarrow \int_{\theta^{\inf}}^{\theta^{\sup}} \left[c^* + \lambda\mu(\theta) - S(c^*, \theta)v(\theta) + U(\theta^{\inf}) + \int_{\theta^{\inf}}^{\theta} S'_\theta(c^*, z)v(z)\, \mathrm{d}z \right] \mathrm{d}F(\theta) = 0.$$

Using the integration by parts formula,

$$\int_{\theta^{\inf}}^{\theta^{\sup}} \int_{\theta^{\inf}}^{\theta} S'_\theta(c^*, z)v(z)\, \mathrm{d}z\, \mathrm{d}F(\theta) = \int_{\theta^{\inf}}^{\theta^{\sup}} \frac{1 - F(\theta)}{f(\theta)} S'_\theta(c^*, \theta)v(\theta)\, \mathrm{d}F(\theta),$$

the feasibility constraint allows us to get

$$U(\theta^{\inf}) = -\int_{\theta^{\inf}}^{\theta^{\sup}} \left[c^* + \lambda\mu(\theta) - S(c^*, \theta)v(\theta) + \frac{1 - F(\theta)}{f(\theta)} S'_\theta(c^*, \theta)v(\theta) \right] \mathrm{d}F(\theta).$$

Finally (28) gives us the indirect utility $\tilde{V}(\theta^{\inf}) = u(c^*, \theta^{\inf}) + U(\theta^{\inf})$ of the least type

$$\tilde{V}(\theta^{\inf}) = u(c^*, \theta^{\inf}) - \int_{\theta^{\inf}}^{\theta^{\sup}} \left[c^* + \lambda\mu(\theta) - S(c^*, \theta)v(\theta) + \frac{1 - F(\theta)}{f(\theta)} S'_\theta(c^*, \theta)v(\theta) \right] \mathrm{d}F(\theta).$$

In the absence of income noise, $\mu(\theta) = v(\theta) = 0$ for all θ, we have

$$\tilde{V}(\theta^{\inf}) = u(c^*, \theta^{\inf}) - \int_{\theta^{\inf}}^{\theta^{\sup}} c^* \,\mathrm{d}F(\theta) = u(c^*, \theta^{\inf}) - c^* = V(\theta^{\inf}).$$

Therefore local income tax randomizations do improve upon the deterministic optimum, $\tilde{V}(\theta^{\inf}) > V(\theta^{\inf})$, if and only if

$$\int_{\theta^{\inf}}^{\theta^{\sup}} \left[\lambda\mu(\theta) - S(c^*, \theta)v(\theta) + \frac{1 - F(\theta)}{f(\theta)} S'_\theta(c^*, \theta)v(\theta) \right] \mathrm{d}F(\theta) < 0.$$

After replacing $S(c^*, \theta)$ and $S'_\theta(c^*, \theta)$ with their expressions in (30) and (33), and using the restriction $\mu(\theta) = v(\theta)$ for all θ, this inequality rewrites as in Proposition 3 below.

Proposition 3. *Consider a menu of lotteries $(\mu(\theta), v(\theta))$ such that $\mu(\theta) = v(\theta)$ for all θ. Let $v(\theta)$ be non-increasing bounded from above. The random menu improves upon the deterministic optimum if and only if*

$$\int_{\theta^{\inf}}^{\theta^{\sup}} \phi(c^*, \theta)v(\theta) \,\mathrm{d}F(\theta) > 0,$$

where

$$\phi(c, \theta) = \left(\frac{1}{c + \theta} - 1 + \frac{m(\theta)}{(c + \theta)^2} \right) - \frac{1}{(c + \theta)^2} \left(\frac{1}{2} + \frac{m(\theta)}{(c + \theta)} \right).$$

An intuition for this condition obtains by considering the total collected tax

$$\int_{\theta^{\inf}}^{\theta^{\sup}} [y(\theta) - c^* - \lambda\mu(\theta)] \,\mathrm{d}F(\theta).$$

Using the definition of $U(\theta) = S(c^*, \theta)v(\theta) - y(\theta)$ and our modeling restriction (27) that the average transfer $\mu(\theta)$ to type θ coincides with $v(\theta)$, this tax can be rewritten as

$$\int_{\theta^{\inf}}^{\theta^{\sup}} [S(c^*, \theta)v(\theta) - U(\theta) - c^* - \lambda v(\theta)] \,\mathrm{d}F(\theta).$$

Let us first maintain $U(\theta)$ at its initial level. A marginal increase $\mathrm{d}v > 0$ in the variance of the after-tax income of types between θ and $\theta + \mathrm{d}\theta$, $\mathrm{d}\theta$ close to 0, yields a change in the total collected tax equal to $[S(c^*,\theta) - \lambda]\, f(\theta)\, \mathrm{d}v\, \mathrm{d}\theta$. Types located between θ and $\theta + \mathrm{d}\theta$ increase their labor supply, which yields an additional amount of tax resources $S(c^*,\theta)f(\theta)\,\mathrm{d}v\,\mathrm{d}\theta > 0$. However a greater noise comes with a greater mean transfer, which costs $\lambda f(\theta)\,\mathrm{d}v\,\mathrm{d}\theta > 0$ to the government budget. This shows that, given $U(\theta)$, the reform would yield more taxes if $S(c^*,\theta) > \lambda$. From (30) and $\lambda > 0$, this is equivalent to

$$\frac{1}{c^*+\theta}\left(1 - \frac{1}{2}\frac{1}{c^*+\theta}\right) > 1,$$

a condition that is never satisfied since $c^* = 1$ and $\theta \geq 0$.

However introducing noise affects $U(\theta)$ through incentives. Indeed, by (32), a necessary condition for incentive compatibility is $U'(\theta) = S'_\theta(c^*,\theta)v(\theta)$ for all θ. To meet incentives the reform thus implies $\mathrm{d}U'(\theta) = S'_\theta(c^*,\theta)\,\mathrm{d}v$. It follows that the sub-utility $U(\theta)$ is also affected: we have $\mathrm{d}U = \mathrm{d}U'(\theta)\,\mathrm{d}\theta = S'_\theta(c^*,\theta)\,\mathrm{d}v\,\mathrm{d}\theta$ for all types above $\theta + \mathrm{d}\theta$. Neglecting the second-order utility change for types between θ and $\theta + \mathrm{d}\theta$, the impact on the total collected tax is $-(1-F(\theta))S'_\theta(c^*,\theta)\,\mathrm{d}v\,\mathrm{d}\theta$. Since $S'_\theta \leq 0$ this is a positive change: random tax perturbations relax incentives, which allows the government to collect additional resources.

Finally the change in tax induced by the introduction of a small noise on the after-tax income is

$$[S(c^*,\theta) - \lambda]\, f(\theta)\,\mathrm{d}v\,\mathrm{d}\theta - (1-F(\theta))S'_\theta(c^*,\theta)\,\mathrm{d}v\,\mathrm{d}\theta,$$

or equivalently,

$$[S(c^*,\theta) - \lambda - m(\theta)S'_\theta(c^*,\theta)]\, f(\theta)\,\mathrm{d}v\,\mathrm{d}\theta.$$

Using the expressions of $S(c^*,\theta)$ and $S'_\theta(c^*,\theta)$ given in (30) and (33), one obtains

$$S(c^*,\theta) - \lambda - m(\theta)S'_\theta(c^*,\theta) = \lambda\phi(c^*,\theta).$$

This shows that $\phi(c^*,\theta)$ actually governs the change in the total collected tax that results from introducing small perturbations on the after-tax income of low type agents, while exploiting the reversal of incentives to reduce the net transfers given to high type agents.

7.4 Application to Generalized Weibull

In the expression of $\phi(c^*,\theta)$ given in Proposition 3, the term into the first bracket is the derivative in after-tax income c of the contribution of type θ to the deterministic objective $V(\theta^{\inf})$ given in (23). This contribution is concave in c. If (25) is satisfied for all θ, its global maximizer, which we denote by $c^{**}(\theta)$, is increasing in θ. Therefore $c^{**}(\theta) < c^*$ for low types. It follows that $\phi(c,\theta)$ evaluated at $c = c^*$ is negative for such types. This shape makes it difficult to get a socially useful income randomization: in the presence of some income noise, the variance $v(\theta)$ has to be positive for low types, and so all the terms $\phi(c^*, \theta)v(\theta)$ that appear in the sum in Proposition 3 are negative for these low types.

The whole sum may however reach a positive value with the generalized Weibull specification. For some threshold type $\bar{\theta}^* \geq \theta^{\inf}$, let

$$v(\theta) = \begin{cases} v > 0 & \text{for } \theta \leq \bar{\theta}^*, \\ 0 & \text{otherwise.} \end{cases}$$

Choose parameters of the generalized Weibull distribution $a = 0.9$, $b = 0.01$ and $s = 5$, the function $\phi(c^*, \theta)f(\theta)$ evaluated at $c^* = 1$ is single-peaked in θ. It is negative for low and high types, but it reaches a positive value at its maximum: the sum

$$\int_{\theta^{\inf}}^{\bar{\theta}^*} \phi(c^*, \theta) \, \mathrm{d}F(\theta)$$

is positive for every value of the threshold $\bar{\theta}^*$ between 5/2 and 20. Thus, by Proposition 3, offering after-tax income lotteries with positive variance v to types below $\bar{\theta}^*$ chosen in this interval and the optimal deterministic contract to the remaining types improves upon the deterministic optimum.

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Appendix: The deterministic optimum

The government is assumed to design a deterministic menu $(c(\theta), y(\theta))$ subject to the feasibility constraint (2) and the incentive constraints

$$\ln(c(\theta)+\theta)-y(\theta) \geq \ln(c(\tau)+\theta)-y(\tau)$$

for all θ and τ. The incentive constraints are satisfied if and only if

$$y'(\theta)=\frac{c'(\theta)}{c(\theta)+\theta} \Leftrightarrow V'(\theta)=\frac{1}{c(\theta)+\theta} \tag{34}$$

and

$$c'(\theta) \leq 0 \tag{35}$$

for all θ.

From (34), indirect utility is increasing with θ, and so type $\theta^{\inf}$ gets the lowest utility. A Rawlsian government thus chooses to maximize $V(\theta^{\inf})$ subject to (2), (34) and (35). Using (2) at equality and (34) the indirect utility of type θ is

$$V(\theta)=V(\theta^{\inf})+\int_{\theta^{\inf}}^{\theta} \frac{1}{c(z)+z}\,\mathrm{d}z, \tag{36}$$

where $V(\theta^{\inf})$ is given in (23).

The optimal schedule is an after-tax income $(c(\theta))$ that maximizes $V(\theta^{\inf})$ subject to the monotonicity condition $c'(\theta) \leq 0$ for all θ. The associated before-tax income then is $y(\theta)=\ln(c(\theta)+\theta)-V(\theta)$.

Let $c^{**}(\theta)$ be the (interior) after-tax income that maximizes the contribution

$$W(c(\theta),\theta)=\ln(c(\theta)+\theta)-c(\theta)-\frac{m(\theta)}{c(\theta)+\theta}$$

of type θ to the social surplus in (23). That is,

$$c^{**}(\theta)+\theta=\frac{1}{2}(1+\sqrt{1+4m(\theta)}). \tag{37}$$

The monotonicity condition (35) binds for all types if $c^{**}(\theta)$ is always increasing. Using (37), this is equivalent to (25).

The use of utility (22) yields a single tax bracket case at the optimum. Indeed, by Theorem 6.1 in Fleming and Rishel (1975), the after-tax income is continuous provided that the deterministic social program does not display multiple maximizers for some θ.

Since $W(c,\theta)$ is concave in c, the program has at most one maximizer on an interval of types such that the second order monotonicity $c'(\theta) \leq 0$ does not bind at the deterministic optimum ($c'(\theta) < 0$). Consider now an interval of types $[\theta_1, \theta_2]$ such that $c'(\theta) = 0$. For such types, $c(\theta) = \bar{c}$ such that

$$c(\theta_1) = c(\theta_2) = \bar{c}$$

satisfies

$$\int_{\theta_1}^{\theta_2} \left[\frac{1}{\bar{c}+\theta} - 1 + \frac{m(\theta)}{(\bar{c}+\theta)^2} \right] \mathrm{d}F(\theta) = 0$$

Use again $W(c,\theta)$ concave in c, so that the integrand is decreasing in $\bar{c}$, to conclude that there is a unique solution $\bar{c}$.

With single tax bracket, the objective (23) simplifies to

$$\int_{\theta^{\inf}}^{\theta^{\sup}} \left[\ln(c+\theta) - c - \frac{m(\theta)}{c+\theta} \right] \mathrm{d}F(\theta) \tag{38}$$

An integration by parts yields

$$\int_{\theta^{\inf}}^{\theta^{\sup}} \ln(c+\theta)\,\mathrm{d}F(\theta) = \ln(c+\theta^{\inf}) + \int_{\theta^{\inf}}^{\theta^{\sup}} \frac{m(\theta)}{c+\theta}\,\mathrm{d}F(\theta).$$

Therefore the objective (38) equals $\ln(c+\theta^{\inf}) - c$. The after-tax income that maximizes this expression is $c^* = 1$ and the corresponding objective is $V(\theta^{\inf}) = -1$.