Dynamic economics in Practice

Monica Costa Dias and Cormac O'Dea

Motivation

- Many economic decisions (e.g. education take-up, savings or investments) are difficult to rationalise in a static setting
- They all involve some trade-off between present costs and future returns, sometimes in uncertain environments
- The existence of some markets (credit, insurance...) hinges on the dynamic nature of some decisions
- Their existence may reinforce the dynamic nature of the decision process

The problem I

Dynamic microeconomic problems are notably difficult to solve

- Very high dimensional
  - Present cost of a decision depends on present circumstances, and these are a consequence of past circumstances and decisions
  - Future returns may also depend on present circumstances and be realised in many periods and in different ways, possibly influencing future decisions
- In most cases, dynamic problems are not tractable analytically
- Possible solution: break the big problem into a sequence of similar smaller problems that we can solve - use Recursive Methods

The problem II

- *Solution we explore:* break the big problem into a sequence of similar smaller problems that we can solve
- This is what a recursive method called Dynamic Programming does
  - Describe the position of the problem at a moment in time: *the state of the world today* – it summarises all the current information relevant for decision-making
  - Where it might be tomorrow: *the state of the world tomorrow*
  - And how the agents *care about tomorrow vis-a-vis today*
- DP allows us to characterise the problem with two functions
  - Transition function: maps the state today into the state tomorrow
  - Choice function: maps the state today into the endogenous choices

This course I

- Gentle and practical introduction to dynamic optimisation
  - Dynamic programming
  - Numerical solution
  - Computational methods

- Main goals
  - Introduce standard tools to study and solve dynamic optimisation problems in microeconomics
  - Demonstrate practically how these tools are used
  - Discuss their comparative advantages
  - Focus on methods and tools that can be easily extended to more general and complex setups

This course II

- Workhorse: the consumption-savings model
  - Interesting per-se: a key model in economics, underlying the permanent income theory and all developments that hinge on it
  - Inherently dynamic
  - Many interesting variations useful to illustrate how to tackle alternative dynamic problems: uncertainty, risk aversion, life-cycle/infinite horizon, habit formation, many choice or state variables, ...
  - Various alternative specifications reflect underlying assumptions about market structure
  - Crucial tool for policy analysis

This course III

- Practical focus
  - Discuss the approaches and procedures we found useful
  - While keeping an eye on efficiency (but it will not be central)
  - We make no attempt to discuss comprehensively the theoretical foundations of the problem or solution
  - *Goal*: to solve increasingly more realistic (but also more complex) models showing methods that can be extended and applied to other settings

Outline of this course

1. The simplest consumption-savings problem: the cake-eating problem
   The problem; Simple example; Existence and uniqueness of solution
2. Introduction to dynamic programming
   Bellman equation; Recursive solution; Optimality conditions; Numerical solution; Practical implementation
3. Life-cycle income process
   Credit markets; Numerical solution; Practical implementation
4. Stochastic optimisation
   Markov processes; Iid income process; Numerical solution; Practical implementation; Autocorrelated income process; Practical implementation
5. Infinite horizon
   The problem; Existence and uniqueness of solution; Simple example; Numerical solution; Practical implementation

The cake-eating problem

Setup and classical solution

The cake-eating problem

Simplest possible life-cycle consumption-savings problem

- Intertemporal problem of a consumer living for T periods and endowed with initial wealth a_1 in period $t = 1$
- Her goal: to allocate the consumption of this wealth over her T periods of life in order to maximise her lifetime wellbeing
- Consumption is divisible: a continuous decision variable
- Any remaining wealth in period t is productive, generating $k(a)$ units of wealth to consume in the future
- No outstanding debts are allowed at the end of life
- And any remaining wealth at the end of life is of no value

Formal model

$$
\begin{aligned}
\max_{(c_1,\ldots,c_T)\in\mathbb{C}^T} \quad & \sum_{t=1}^{T} \beta^{t-1} u(c_t) \\
\text{s.t} \quad & a_{t+1} = k(a_t - c_t) \text{ for } t = 1, \ldots, T \\
& a_{T+1} \geq 0 \\
& a_1 \, (\in \mathbb{A}) \text{ given}
\end{aligned}
$$

- Per-period wellbeing u: increasing in consumption
- Consumption: choice variable, with domain $\mathbb{C}$ (here $\mathbb{R}_0^+$ or $\mathbb{R}^+$, depending on u)
- Assets is the state variable, with domain $\mathbb{A}$ (here $\mathbb{R}_0^+$ or $\mathbb{R}^+$)
- k: law of motion for assets, a positive and increasing function in $\mathbb{A}$

$$k\left(a_t - c_t\right) = R\left(a_t - c_t\right) \quad \text{where } R = 1 + r \text{ is the interest factor}$$

Classical solution

- Objective function is $\mathcal{C}^1$ (continuously differentiable): interior optimum satisfies foc
- Classical solution: attack problem directly by solving all its foc's
- Useful to write model restrictions more compactly by noting that the law of motion for assets together with the initial condition imply

$$a_{T+1} = R^T a_1 - \sum_{t=1}^{T} R^{T-t+1} c_t$$

- Therefore, the consumer's problem for a given $a_1 \geq 0$ is

$$\max_{(c_1,\ldots,c_T)\in(\mathbb{C})^T} \sum_{t=1}^{T} \beta^{t-1} u(c_t) \qquad \text{s.t} \ \sum_{t=1}^{T} R^{1-t} c_t \leq a_1$$

Classical solution: Euler equation I

- Lagrangian for this problem

$$\mathcal{L} = \sum_{t=1}^{T} \beta^{t-1} u(c_t) - \lambda \left(\sum_{t=1}^{T} R^{1-t} c_t - a_1 \right)$$

- With necessary foc's with respect to c_t, for $t = 1, \ldots, T$:

$$\beta^{t-1} u'(c_t) = \lambda R^{1-t}$$

- Putting together two subsequent conditions yields

$$u'(c_t) = \beta R u'(c_{t+1}) \quad \text{for } t = 1, \ldots, T-1 \tag{1}$$

- These are the Euler equations for this problem

Classical solution: Euler equation II

$$u'(c_t) = \beta R u'(c_{t+1}) \quad \text{for } t = 1, \ldots, T-1$$

- Euler equation: establishes relationship between consumption in subsequent periods
- But not the consumption level
- For that we need the budget constraint
- The Kuhn-Tucker conditions do just that

Classical solution: Kuhn-Tucker conditions

- The Kuhn-Tucker conditions for this problem:

$$\lambda \left(\sum_{t=1}^{T} R^{1-t} c_t - a_1 \right) = 0, \quad \lambda \geq 0, \quad \sum_{t=1}^{T} R^{1-t} c_t \leq a_1$$

- If u strictly increasing ($u' > 0$):
  - $\lambda > 0$: +ve marginal value of relaxing the budget constraint
  - $\sum_{t=1,\ldots,T} R^{1-t} c_t = a_1$: consumer better off by consuming all a_1
  - Then

$$a_{T+1} = 0 \tag{2}$$

- Together, the T conditions (**??**) and (**??**) determine the T interior optimal consumption choices

Corner solutions

- Up to here we assumed that the solution is interior
- The Euler conditions allowing for corner solutions are

$$\begin{aligned} & u'(c_t) \leq \beta R u'(c_{t+1}) \quad \text{for the possibility of } c_t = 0 \\ \text{or} \quad & u'(c_t) \geq \beta R u'(c_{t+1}) \quad \text{for the possibility of } c_t = a_t \end{aligned}$$

- Typical choices of utility functions are continuously differentiable and monotonically increasing in $\mathbb{R}^+$, with the additional following property:

$$\lim_{c_t \to 0^+} u(c_t) = -\infty \quad \text{and} \quad \lim_{c_t \to 0^+} u'(c_t) = +\infty$$

In this case a solution, if it exists, is interior

The cake-eating problem

Simple example: CRRA utility

CRRA utility

- A convenient and popular specification of the utility function ($\gamma > 0$)

$$u(c) = \frac{c^{1-\gamma}}{1-\gamma}$$

  γ^{-1} is the elasticity of intertemporal substitution

- It is generally accepted that $\gamma \geq 1$, in which case, for $c \in \mathbb{R}^+$

$$\begin{array}{lll} u(c) < 0, & \lim_{c\to 0} u(c) = -\infty, & \lim_{c\to +\infty} u(c) = 0 \\ u'(c) > 0, & \lim_{c\to 0} u'(c) = +\infty, & \lim_{c\to +\infty} u'(c) = 0 \end{array}$$

CRRA utility: solution I

- The problem is

$$\max_{(c_1,\dots,c_T)\in(\mathbb{R}^+)^T} \sum_{t=1}^{T} \beta^{t-1}\frac{c_t^{1-\gamma}}{1-\gamma} \qquad \text{s.t} \sum_{t=1}^{T} R^{1-t}c_t \leq a_1$$

- Euler equations:

$$c_t^{-\gamma} = \beta R c_{t+1}^{-\gamma} \quad \Rightarrow \quad c_t = (\beta R)^{-\frac{1}{\gamma}} c_{t+1} \quad \text{for } t = 1, \dots, T-1$$

- By successive substitution:

$$c_t = (\beta R)^{\frac{t-1}{\gamma}} c_1$$

CRRA utility: solution II

- The budget constraint and optimality condition imply

$$
\begin{aligned}
a_1 &= \sum_{t=1,\dots,T} R^{1-t} c_t \\
&= c_1 \sum_{t=1,\dots,T} \left(\beta^{\frac{1}{\gamma}} R^{\frac{1-\gamma}{\gamma}}\right)^{t-1} \\
&= c_1 \sum_{t=1,\dots,T} \alpha^{t-1} \quad \text{where} \quad \alpha = \beta^{\frac{1}{\gamma}} R^{\frac{1-\gamma}{\gamma}}
\end{aligned}
$$

- The solution for $t = 1, \dots, T$:

$$
c_1 = \frac{1-\alpha}{1-\alpha^T} a_1 \qquad \text{and} \qquad c_t = \frac{1-\alpha}{1-\alpha^T} (\beta R)^{\frac{t-1}{\gamma}} a_1
$$

CRRA utility: solution III

In general, if the optimisation problem starts at time t as follows

$$\max_{(c_t,\ldots,c_T)\in(\mathbb{R}^+)^{T-t+1}} \sum_{\tau=t}^{T} \beta^{\tau-t}\frac{c_\tau^{1-\gamma}}{1-\gamma} \qquad \text{s.t} \ \sum_{\tau=t}^{T} R^{\tau-t}c_\tau \ \leq \ a_\tau$$

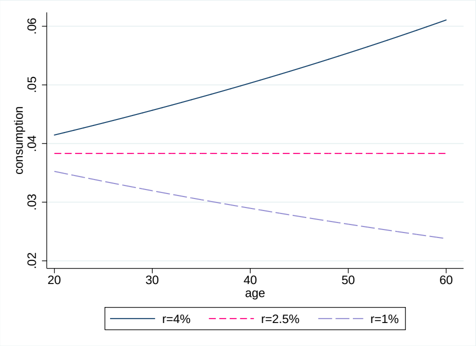
the solution for c_t is

$$c_t \ = \ \frac{1-\alpha}{1-\alpha^{T-t+1}}a_t$$

This is the *consumption function*, a linear function of assets if utility is CRRA

CRRA utility: consumption over the life-cycle

βR determines the profile of the solution: $c_t = \frac{1-\alpha}{1-\alpha^T}(\beta R)^{\frac{t-1}{\gamma}} a_1$



$\beta = 1.025^{-1}$ and initial assets are $a_{20} = 1$.

The cake-eating problem

Existence and uniqueness of solution

When can the existence of the optimum be guaranteed?

- Feasibility set: space of choices satisfying the problem constraints

$$\mathcal{C}_{1:T}(a_1) \quad = \quad \left\{ (c_1, \ldots, c_T) \in \mathbb{C}^T : \sum_{t=1,\ldots,T} R^{1-t} c_t \leq a_1 \right\}$$

  where typically $\mathbb{C} = \mathbb{R}^+$

- Apply Weierstrass theorem to ensure existence of solution:

  Let $u : \mathbb{C} \to \mathbb{R}$ be continuous and suppose $\mathcal{C}_{1:T}(a_1) \subset \mathbb{C}^T$ is non-empty and compact. Then the consumer's problem

$$\max_{(c_1,\ldots,c_T)\in\mathcal{C}_{1:T}(a_1)} \sum_{t=1,\ldots,T} \beta^{t-1} u(c_t)$$

  has a solution in $\mathcal{C}_{1:T}(a_1)$ for any $a_1 \in \mathbb{A}$.

When is the optimum interior and unique?

- Typical consumer's problem: u is strictly increasing, concave and $\mathcal{C}^1$
  - Then the sum of per-period utilities is also strictly increasing, concave and $\mathcal{C}^1$
- Also assume that the feasibility set $\mathcal{C}_{1:T}(a_1)$ is non-empty and compact
- Under these conditions the solution is unique
- It is also interior ($T > 1$)
- But if we had a convex u: corner solution

Dynamic programming

The Bellman equation

Dynamic programming

- Dynamic programming splits the big problem into smaller problems that are of similar structure *and* easier to solve
- The trick is to find the limited set of variables that completely describe the decision problem in each period – the state
- Then the solution of these problems over a small state-space determines a set of policy functions: optimal consumption is $h_t(a_t)$ for $t = 1, \dots T$
- DP returns a general solution: it solves the entire family of problems of the same type
- The specific solution to our problem can be constructed recursively, by iterating

$$\begin{aligned} c_t &= h_t(a_t) \\ a_{t+1} &= R(a_t - c_t) \end{aligned}$$

starting from the given a_1

Problem specification I

- In our problem, the level of assets at the start of period t summarises all the information needed to solve for consumption
- The feasibility set at time t for the sequence of present and future consumption choices given $a_t \in \mathbb{A}$ is

$$\mathcal{C}_{t:T}(a_t) = \left\{ (c_t, \ldots, c_T) \in \mathbb{C}^{T-t+1} : \sum_{\tau=t,\ldots,T} R^{t-\tau} c_\tau \leq a_t \right\}$$

- If consumption must be positive in every period, then $\mathbb{C} = \mathbb{A} = \mathbb{R}^+$ and the feasibility set at time t is

$$\mathcal{C}_t(a_t) = \begin{cases} \{c_t > 0 : \ a_{t+1} = R\,(a_t - c_t) > 0\} & \text{if } t < T \\ \{c_t > 0 : \ a_{t+1} = R\,(a_t - c_t) \geq 0\} & \text{if } t = T \end{cases}$$

Problem specification II

- The problem of a consumer with assets a_t at time t is

$$V_t(a_t) = \max_{(c_t,\ldots,c_T)\in\mathcal{C}_{t:T}(a_t)} \sum_{\tau=t,\ldots,T} \beta^{\tau-t} u(c_\tau)$$

- V_t is the value function
  - Indirect lifetime utility: measures max utility that assets a_t can deliver
  - It is a function of a_t alone
  - Dependence on a_t arises through the feasibility set

Problem specification III

The value function can be defined recursively

$$
\begin{aligned}
V_t(a_t) &= \max_{(c_t,\ldots,c_T)\in\mathcal{C}_{t:T}(a_t)} \sum_{\tau=t,\ldots,T} \beta^{\tau-t} u(c_\tau) \\
&= \max_{c_t\in\mathcal{C}_t(a_t)} \left\{ u(c_t) + \beta \left[\underbrace{\max_{(c_{t+\mathbf{1}},\ldots,c_T)\in\mathcal{C}_{t+\mathbf{1}:T}(a_{t+\mathbf{1}})} \sum_{\tau=t+1}^{T} \beta^{\tau-(t+1)} u(c_\tau)}_{V_{t+\mathbf{1}}(a_{t+\mathbf{1}})} \right] \right\} \\
&= \max_{c_t\in\mathcal{C}_t(a_t)} \{ u(c_t) + \beta V_{t+1}\left(R[a_t - c_t]\right) \}
\end{aligned}
$$

The Bellman equation I

$$V_t(a_t) = \max_{c_t \in \mathcal{C}_t(a_t)} \{u(c_t) + \beta V_{t+1}(R[a_t - c_t])\}$$

- This is a functional equation: recursive formulation
- Breaks the large lifecycle problem in smaller static problems
  - Key: memoryless process depends only on the value of state variables at the time of decision
- Principle of Optimality: if the consumer behaves optimally in the future, all that matters for the solution at time t is the decision of how much to consume today
- V_{t+1} exists (by recursion) but is unknown!

The Bellman equation II

Often useful to reformulate the problem in terms of savings decisions

- Define the payoff function as

$$f\left(a_t, a_{t+1}\right) = u\left(a_t - \frac{a_{t+1}}{R}\right) = u\left(c_t\right)$$

- Then the consumption/savings problem is equivalently specified as

$$V_t(a_t) = \max_{a_{t+1} \in \mathcal{D}_t(a_t)} \left\{ f\left(a_t, a_{t+1}\right) + \beta V_{t+1}\left(a_{t+1}\right) \right\}$$

where the feasibility set at time t (for $\mathbb{C} = \mathbb{A} = \mathbb{R}^+$)

$$\mathcal{D}_t(a_t) = \begin{cases} \left\{a_{t+1} > 0 : \; a_t - a_{t+1}R^{-1} > 0, \right\} & \text{if } t < T \\ \left\{a_{t+1} \geq 0 : \; a_t - a_{t+1}R^{-1} > 0 \right\} & \text{if } t = T \end{cases}$$

The Bellman equation III

- The solution is

$$g_t(a_t) \quad = \quad \underset{a_{t+1} \in \mathcal{D}_t(a_t)}{\arg\max} \; \{ f(a_t, a_{t+1}) + \beta V_{t+1}(a_{t+1}) \}$$

- Exists and is unique under the conditions discussed earlier:
  - f real-valued, strictly increasing (decreasing) in the first (second) argument, concave and $\mathcal{C}^1$ in both arguments
  - $\mathcal{D}$ is non-empty and compact
- Under these conditions g is also continuous
- Moreover, V inherits some of the properties of f
  - continuity, monotonicity and concavity
  - differentiability at points $a \in \mathbb{A}$ in which the solution is interior

Dynamic programming

Recursive solution

Recursive solution

$$V_t(a_t) = \max_{a_{t+1} \in \mathcal{D}_t(a_t)} \{f(a_t, a_{t+1}) + \beta V_{t+1}(a_{t+1})\}$$

Key insight of dynamic programming: the unknown V can be pinned down by backward induction

- This highlights the usefulness of the Bellman equation
- And inspires the numerical strategy to solve models with no closed-form solution

Last period

Solution strategy: start from period T and move backwards as the future value function, the continuation value, is determined

- The problem in the last period is

$$V_T(a_T) = \max_{a_{T+1} \in \mathcal{D}_T(a_T)} \{f(a_T, a_{T+1})\}$$

where $\mathcal{D}_T(a_T) = [0, Ra_T]$

- The solution is (for any $a_T \in \mathbb{A}$)

$$g_T(a_T) = 0 \quad \text{with value} \quad V_T(a_T) = f(a_T, 0) = u(a_T)$$

Last but one period

- Since $V_T(a_T) = u(a_T)$, the problem at $T-1$ is known

$$V_{T-1}(a_{T-1}) \quad = \quad \max_{a_T \in \mathcal{D}_{T-1}(a_{T-1})} \{f(a_{T-1}, a_T) + \beta V_T(a_T)\}$$

- Under differentiability of the maximising function, an interior optimum satisfies the foc's (for any $a_{T-1} \in \mathbb{A}$

$$g_{T-1}(a_{T-1}) \quad \text{is the solution to} \quad f_2\left(a_{T-1}, a_T\right) + \beta V_T'(a_T) = 0$$

- So the value function at $T-1$ is (for each $a_{T-1} \in \mathbb{A}$)

$$V_{T-1}(a_{T-1}) \quad = \quad f\left(a_{T-1}, g_{T-1}(a_{T-1})\right) + \beta V_T\left(g_{T-1}(a_{T-1})\right)$$

Period t

Move backwards in similar steps

- Once the value function for period $t+1$ has been determined, solve (for each $a_t \in \mathbb{A}$)

$$g_t(a_t) \quad = \quad \underset{a_{t+\mathbf{1}} \in \mathcal{D}_t(a_t)}{\arg\max} \; \{f(a_t, a_{t+1}) + \beta V_{t+1}(a_{t+1})\}$$

- The solution can then be used to build V_t (for each $a_t \in \mathbb{A}$):

$$V_t(a_t) \quad = \quad f\left(a_t, g_t(a_t)\right) + \beta V_{t+1}\left(g_t(a_t)\right)$$

Solution to our specific problem

- The specific problem we are interested in is fully characterised by the initial condition, a_1
- To construct the solution, we use the policy functions g_t and iterate, for $t = 1, \ldots, T$

$$c_t = a_t - R^{-1} g_t(a_t) \quad \text{and} \quad a_{t+1} = g_t(a_t)$$

Dynamic programming

Optimality conditions

Optimality conditions I

- The typical problem in economics assumes that the utility function is strictly increasing, concave and continuously differentiable (in consumption), and that the feasibility space is closed and bounded
- Under these conditions the solution is unique and V is differentiable
- And the first order conditions are necessary and sufficient for an interior optimum

Optimality conditions II

- The problem at time t is

$$V_t(a_t) = \max_{a_{t+1} \in \mathcal{D}_t(a_t)} \{f(a_t, a_{t+1}) + \beta V_{t+1}(a_{t+1})\}$$

- The foc at time t is

$$f_2(a_t, a_{t+1}) + \beta V'_{t+1}(a_{t+1}) = 0$$

- Use the envelope condition to workout $V'_{t+1}(a_{t+1})$

$$\begin{aligned}
V'_t(a_t) &= f_1(a_t, a_{t+1}) + f_2(a_t, a_{t+1}) \frac{\partial a_t + 1}{\partial a_t} + \beta V'_{t+1}(a_{t+1}) \frac{\partial a_t + 1}{\partial a_t} \\
&= f_1(a_t, a_{t+1}) + \underbrace{\left[f_2(a_t, a_{t+1}) + \beta V'_{t+1}(a_{t+1})\right]}_{\text{foc at } t} \frac{\partial a_t + 1}{\partial a_t} \\
&= f_1(a_t, a_{t+1}) = u'(h_t(a_t))
\end{aligned}$$

Optimality conditions III

Put the foc together with the envelope condition to get the Euler equation

$$
\begin{aligned}
f_2(a_t, a_{t+1}) + \beta f_1(a_{t+1}, a_{t+2}) &= 0 \\
\Leftrightarrow \quad u'(c_t) &= \beta R u'(c_{t+1})
\end{aligned}
$$

since: $u(c_t) = f(a_t, a_{t+1}) = u\left(a_t - \frac{a_{t+1}}{R}\right)$

and so: $f_1(a_t, a_{t+1}) = u'(c_t)$ and $f_2(a_t, a_{t+1}) = -\frac{u'(c_t)}{R}$

Dynamic programming

Numerical solution

Numerical solution

- The cake-eating problem is easy to solve on the paper
- But it is an instructive example to play with numerically
  - Sophisticated enough to require most of the numerical tricks used in more complicated models
  - But easy enough to keep the discussion simple
  - Can be used to demonstrate the comparative advantages of various numerical procedures since the solution is known!

Computers do not known infinity

1. Model specification
   - CRRA utility is great to ensure that consumers avoid getting close to zero consumption
   - The same does not hold for computational solutions: extreme values cause the routine to crash
     $\Rightarrow$ Bound solution space to its relevant parts to avoid problems
2. Discretise state space
   - Select grid in assets $A = \{a^i\}_{i=1,\ldots,n_a}$
   - Solve problem only for points in the grid
   - Approximate unknown functions numerically outside the grid

Algorithm for recursive solution

1. Parameterise model and select grid in assets: $\left\{a^i\right\}_{i=1,\ldots,n_a}$
2. Choose stopping criterion $\epsilon > 0$
3. Store $V_{T+1}\left(a^i\right) = 0$ for all $i = 1, \ldots, n_a$
4. Loop over t backwards: $t = T, \ldots, 1$

   For each $i = 1, \ldots, n_a$

   4.1 Compute $g_t^i = \arg\max_{a_{t+1} \in \mathcal{D}_t(a^i)} \left\{u\left(a^i - \frac{a_{t+1}}{R}\right) + \beta \widetilde{V}_{t+1}\left(a_{t+1}\right)\right\}$

   4.2 Compute $V_t^i = u\left(a^i - \frac{g_t^i}{R}\right) + \beta \widetilde{V}_{t+1}\left(g_t^i\right)$

   4.3 Approximate V_t over its entire domain to get $\widetilde{V}_t$ and store it
   This step is optional: can be done directly in step 4.1 or skipped altogether, depending on the solution method - more to follow

Solution at each point

- Step 4.1 is the (computationally) heavy part of the solution algorithm
- There are two main ways of finding the optimum g_t^i
  - Use a search algorithm to look for the value of savings a_{t+1} that maximise $V_t(a_t)$

    This is the procedure implicit in the algorithm we presented
  - Or look for the root of the Euler equation $u'(c_t) = \beta R V'(a_{t+1})$ vspace0.1cm *We will discuss this solution later*

Solution at each point using the foc: a trick I

- Useful trick under CRRA: speed up and improve accuracy of solution
- The Euler equation is

$$c_t^{-\gamma} = \beta R V'_{t+1}(a_{t+1}) \quad \Leftrightarrow \quad c_t = (\beta R)^{-1/\gamma} \left[V'_{t+1}(a_{t+1}) \right]^{-1/\gamma}$$

- But since (envelope condition)

$$V'_{t+1}(a_{t+1}) = u'\left(h_{t+1}(a_{t+1})\right) = \left(a_{t+1} - \frac{g_{t+1}(a_{t+1})}{R} \right)^{-\gamma}$$

- The solution is the level of savings a_{t+1} that satisfies

$$\underbrace{a_t - \frac{a_{t+1}}{R}}_{c_t} = (\beta R)^{-1/\gamma} \underbrace{\left[a_{t+1} - \frac{g_{t+1}(a_{t+1})}{R} \right]}_{c_{t+1}}$$

Solution at each point using the foc: a trick II

$$\underbrace{a_t - \frac{a_{t+1}}{R}}_{c_t} = (\beta R)^{-1/\gamma} \underbrace{\left[a_{t+1} - \frac{g_{t+1}(a_{t+1})}{R}\right]}_{c_{t+1}} = (\beta R)^{-1/\gamma} h_{t+1}(a_{t+1})$$

- This is a linear (in a_{t+1}) equation in non-stochastic problems
- More generally, the policy function h is typically not very non-linear
- So all we need is to:
  1. Store $h_t(a^i)$ after solving consumers problem at time t
  2. "Connect the points" to approximate function h and obtain the solution over the entire domain: Linear Interpolation
- Notice that V is not needed to solve the problem using the foc

Approximating the value function

- A bad idea: to rely on simple (linear) approximations of V to solve model as V can be highly non-linear
- But one may still need the value function, even when relying on the foc for the solution:
  - to study the value of different policy interventions
  - or attitudes towards risk once uncertainty is considered
- Two alternatives to approximate V
  - More reliable approximation method: shape-preserving splines
  - Reduce non-linearity by applying selected transformation, then approximate by linear interpolation
    For a CRRA utility:

$$\Psi_t(a_t) = [(1-\gamma)V_t(a_t)]^{\frac{1}{1-\gamma}}$$

Practical session 1

Income process

Add income process

- Just adding an income process does not much change the lifecycle problem
- But raises interesting issues of how to deal with the credit markets
- Suppose the consumer has a stream of income over time

$$y_t = w(a_t, t)$$

- For the moment, suppose $\{y_t\}_{t=1,\dots,T}$ is known by the consumer from time $t = 1$

Income process

Credit Markets

Functioning credit markets I

- If credit markets are complete, the consumer may borrow to bring income forward
  - Assets at time t can be negative
  - Borrowing limited by ability to repay
  - Domain of possible values of assets changes over time, depending on time left to repay debts and terminal condition
- The problem of the consumer at time t for assets a_t

$$
\begin{aligned}
V_t(a_t, y_t) \quad &= \quad \max_{a_{t+1}} \{f(a_t, y_t, a_{t+1}) + \beta V_{t+1}(a_{t+1}, y_{t+1})\} \\
\text{s.t.} \quad & \quad a_{t+1} = R(a_t + y_t - c_t) \\
& \quad y_{t+1} = w(a_{t+1}, t+1) \\
& \quad c_t > 0 \quad \text{and} \quad a_{T+1} \geq 0
\end{aligned}
$$

Functioning credit markets II

- The feasibility space at time $t < T$ is

$$\mathcal{D}_t(a_t, y_t) = \left\{ a_{t+1} : \underbrace{a_t + y_t - \frac{a_{t+1}}{R}}_{c_t} > 0,\ a_{t+1} + \sum_{\tau=t+1}^{T} R^{(t+1)-\tau} y_\tau > 0 \right\}$$
$$= \left(-\sum_{\tau=t+1}^{T} R^{(t+1)-\tau} y_\tau,\ R\left(a_t + y_t\right) \right)$$

- At time T

$$\mathcal{D}_T(a_T, y_T) = [0,\ R\left(a_T + y_T\right))$$

Functioning credit markets III

- The compact specification of the problem is

$$
\begin{aligned}
V_t(a_t, y_t) &= \max_{a_{t+1} \in \mathcal{D}_t(a_t, y_t)} \{f(a_t, y_t, a_{t+1}) + \beta V_{t+1}(a_{t+1}, y_{t+1})\} \\
\text{s.t.} & \quad y_t = w(a_t, t) \quad \text{for all } t
\end{aligned}
$$

- Foc is Euler equation $u'(c_t) = \beta R u'(c_{t+1})$
- The state space is now 2-dimensional
  - Although it is easy to reduce to 1 dimension in this case by noting that $a_{t+1} = R(a_t + w(a_t, t) - c_t)$
  - Computation-wise, reducing the dimensionality of the state space is the most time-saving procedure

Simple example: CRRA utility

- With CRRA utility the Euler equation implies $c_t = (\beta R)^{\frac{t-1}{\gamma}} c_1$
- The value of total lifetime wealth at $t = 1$ is

$$W \ = \ a_1 + \sum_{t=1,\ldots,T} R^{1-t} y_t$$

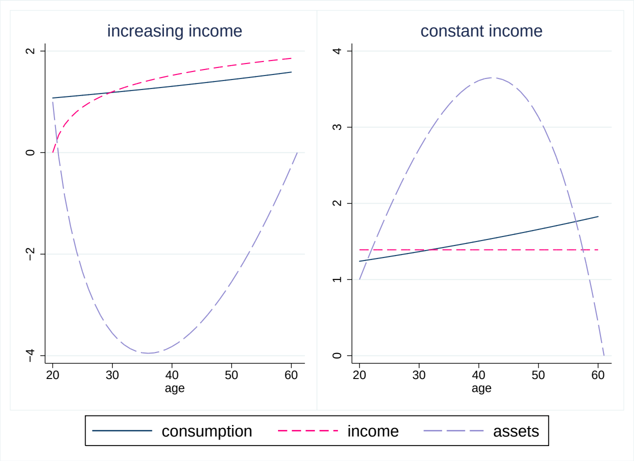
- Total consumption is

$$C \ = \ \sum_{t=1,\ldots,T} R^{1-t} c_t \ = \ \sum_{t=1,\ldots,T} \left(\beta R^{1-\gamma}\right)^{\frac{t-1}{\gamma}} c_1$$

- Yielding, for $t = 1, \ldots, T$

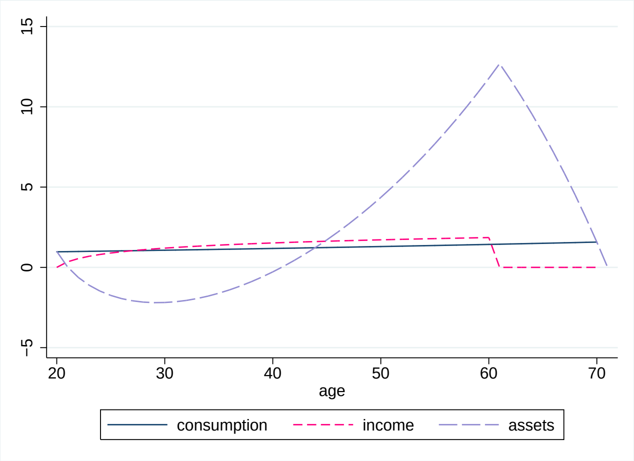
$$c_t \ = \ (\beta R)^{\frac{t-1}{\gamma}} \frac{1-\alpha}{1-\alpha^T} W \qquad \text{where} \ \ \alpha = \beta^{\frac{1}{\gamma}} R^{\frac{1-\gamma}{\gamma}}$$

CRRA utility: profiles for a patient consumer



$r = 4\%$ and $\beta = 1.025^{-1}$. Initial assets are $a_1 = 1$. Income profiles as plotted.

CRRA utility: introducing retirement



$r = 4\%$ and $\beta = 1.025^{-1}$. Initial assets are $a_1 = 1$. Income profiles as plotted.

Credit constraints I

- If credit is rationed, the consumer may be willing to consume more than she can afford in the short term
- In the absence of credit, the feasibility set is restricted to

$$\mathcal{D}_t(a_t, y_t) = \left\{ a_{t+1} : \ a_t + y_t - \frac{a_{t+1}}{R} > 0, \ a_{t+1} \geq 0 \right\}$$

- This implies that the consumer's best choice may be a corner solution

Credit constraints II

- The problem of the consumer at time t for assets a_t is now

$$
\begin{aligned}
V_t(a_t, y_t) &= \max_{a_{t+1}} \{f(a_t, y_t, a_{t+1}) + \beta V_{t+1}(a_{t+1}, y_{t+1})\} \\
\text{s.t.} \quad & a_{t+1} = R(a_t + y_t - c_t) \\
& y_{t+1} = w(a_{t+1}, t+1) \\
& c_t > 0 \quad \text{and} \quad a_{t+1} \geq 0
\end{aligned}
$$

- There are T inequality restrictions in assets now, so we have T first order and Kuhn Tucker conditions:

$$
\begin{aligned}
& f_3(a_t, y_t, a_{t+1}) + \beta f_1(a_{t+1}, y_{t+1}, a_{t+2}) = \lambda_t \\
& \lambda_t a_{t+1} = 0, \quad \lambda_t \geq 0, \quad a_{t+1} \geq 0
\end{aligned}
\qquad \text{for } t = t = 1, \ldots, T-1
$$

$$
a_{T+1} = 0 \qquad \text{for } t = T
$$

Credit constraints III

The solution is

$$c_t = \min\left\{a_t + y_t,\ \text{root of } u'(c_t) = \beta R u'(c_{t+1})\right\}$$

or

$$a_{t+1} = \max\left\{0,\ \text{root of } f_3(a_t, y_t, a_{t+1}) + \beta f_1(a_{t+1}, y_{t+1}, a_{t+2}) = 0\right\}$$

Income process

Numerical solution

Solution algorithm

The recursive solution in practice: almost exactly as before

1. Parameterise model and select grids in a_t: $\left\{a_t^i\right\}_{i=1,\dots,n_a}$
2. Choose stopping criterion $\epsilon > 0$
3. Store $V_{T+1}\left(a_{T+1}^i\right) = 0$ for all $i = 1, \dots, n_a$
4. Loop over t backwards: $t = T, \dots, 1$

   For each $i = 1, \dots, n_a$

   4.1 Compute $g_t^i = \underset{a_{t+1} \in \mathcal{D}_t(a_t^i)}{\arg\max} \left\{ u\left(a_t^i + w(a_t^i, t) - \frac{a_{t+1}}{R}\right) + \beta \widetilde{V}_{t+1}\left(a_{t+1}\right) \right\}$

   4.2 Compute $V_t^i = u\left(a_t^i + w(a_t^i, t) - \frac{g_t^i}{R}\right) + \beta \widetilde{V}_{t+1}\left(g_t^i\right)$

Computational solution: additional issues

1. Dimension of state space: reduce to 1 in solution

$$a_{t+1} = R(a_t + w(a_t, t) - c_t)$$

2. Positive consumption: may be tricky to ensure with approximated functions $\Rightarrow$ impose minimum consumption $c_{min} > 0$

3. Functioning credit markets: grid in assets changes over time

   - Lower bound at t ensures debt can be repaid and c_{min} is affordable

   $$a_t + \sum_{\tau=t\ldots,T} R^{t-\tau} y_\tau \geq \sum_{\tau=t\ldots,T} R^{t-\tau} c_{min}$$

   - Upper bound at t reached if consumes c_{min} in all periods to t

   $$a_t \leq R^{t-1} a_1 + \sum_{\tau=1\ldots,t-1} R^{t-\tau} (y_\tau - c_{min})$$

Practical session 2

Stochastic optimisation

Stochastic problems

- Most interesting problems in economics involve some sort of uninsurable risk
- The solution to the dynamic problem will depend crucially on
  1. how much risk consumers face
  2. their attitudes towards risk
- We consider a stochastic income process to formalise uncertainty
- And do so in a parsimonious way, using Markov processes

Stochastic optimisation

Markov processes

Super brief introduction to stochastic Markov processes I

Stochastic process: sequence $\{y_t\}_{t=1,\ldots}$ of random variables/vectors

The Markov property

- Suppose $\{y_t\}_{t=1,2,\ldots}$ is defined on the support $\mathbb{Y}$
- Then $\{y_t\}$ *satisfies the Markov property* if, for all $y \in \mathbb{Y}$

$$\text{Prob}\left(y_{t+1} = y \mid y_t, \ldots, y_1\right) = \text{Prob}\left(y_{t+1} = y \mid y_t\right) \quad \text{for discrete } \mathbb{Y}$$

$$\text{Prob}\left(y_{t+1} < y \mid y_t, \ldots, y_1\right) = \text{Prob}\left(y_{t+1} < y \mid y_t\right) \quad \text{for continuous } \mathbb{Y}$$

Super brief introduction to stochastic Markov processes II

- The conditional probabilities are known as the transition function

$$Q_t(y_t, y_{t+1}) = \text{Prob}(y_{t+1} \mid y_t)$$

- Time-invariant process: $Q_t(y_t, y_{t+1}) = Q(y_t, y_{t+1})$

- $Q : \mathbb{Y} \times \mathbb{Y} \to [0,1]$ is a transition function if $Q(y_t, y)$ is a pdf:

  For each $y_t \in \mathbb{Y}$

$$Q(y_t, y) \geq 0 \quad \text{for all } y \in \mathbb{Y}$$

$$\text{and} \quad \int_{\mathbb{Y}} Q(y_t, y)\, dy = 1$$

Super brief introduction to stochastic Markov processes III

- Markov process: stochastic process satisfying the Markov property
- Characterised by 3 objects
  - the domain $\mathbb{Y}$
  - the transition function Q
  - the distribution of the initial value y_1
- These fully characterise the joint and marginal distributions of y at all points in time

Super brief introduction to stochastic Markov processes IV

- The unconditional distribution of y_t can be obtained iteratively
- Let π_{t-1} be the pdf of y at time $t-1$. Then, if π_{t-1} is known

$$\pi_t(y_t) \quad = \quad \int_{y \in \mathbb{Y}} Q(y, y_t)\pi_{t-1}(y)\, dy$$

  where π_t be the pdf of y at time t

- A Markov process is stationary if $\pi_t(y) = \pi_{t'}(y) = \pi(y)$
- In this case, π is the *fixed point* in the functional equation

$$\pi(y_t) \quad = \quad \int_{y \in \mathbb{Y}} Q(y, y_t)\pi(y)\, dy$$

Stochastic optimisation

Iid income process

Memoryless income process with discrete support

- Take a discrete income process $y_t \in \mathbb{Y} = \left\{y^1, \ldots, y^n\right\}$
- For a memoryless problem, the transition function equals the unconditional pdf:

$$\pi^i \;=\; \text{Prob}\left(y_t = y^i\right) \;=\; Q\left(y, y^i\right) \text{ for each } i = 1, \ldots, n$$

- The consumer's problem is

$$\begin{aligned} V_t\left(a_t, y_t\right) \quad &= \quad \max_{a_{t+1} \in \mathcal{D}_t(a_t, y_t)} \left\{ f(a_t, y_t, a_{t+1}) + \beta \sum_{y^i \in \mathbb{Y}} V_{t+1}\left(a_{t+1}, y^i\right) \pi^i \right\} \\ \text{s.t.} \quad & \quad y_t \text{ is a rv with pdf } \pi \end{aligned}$$

- The problem is setup as a Markov process: (a_{t+1}, y_{t+1}) depends only on (a_t, y_t)

Memoryless income process with continuous support

- The problem is

$$
\begin{aligned}
& V_t\left(a_t, y_t\right) \\
= & \max_{a_{t+1} \in \mathcal{D}_t(a_t, y_t)} \left\{ f(a_t, y_t, a_{t+1}) + \beta \int_{y \in \mathbb{Y}} V_{t+1}\left(a_{t+1}, y\right) \pi(y)\, dy \right\}
\end{aligned}
$$

- Feasibility set: savings choices ensuring positive consumption is affordable even in *worst possible scenario*

$$
\begin{aligned}
& \mathcal{D}_t(a_t, y_t) \\
= & \left\{ a_{t+1} : a_t + y_t - \frac{a_{t+1}}{R} > 0, \quad a_{t+1} + \sum_{\tau=t+1}^{T} R^{(t+1)-\tau} y_{min} > 0 \right\}
\end{aligned}
$$

Support and feasibility set in practice

- Feasibility set for a_{t+1} is $\mathcal{D}_t(a_t, y_t)$
  - Set of possible choices a_{t+1} given current value of state variables
  - Computational implementation: optimal savings chosen in $\mathcal{D}_t(a_t, y_t)$
- Support of a_{t+1} is $\mathbb{A}_{t+1}$
  - Range of all possible values of a_{t+1}, independently of current value of state variables
  - Computational implementation: grid in a_{t+1} drawn to represent $\mathbb{A}_{t+1}$
- Clearly $\mathcal{D}_t(a_t, y_t) \subseteq \mathbb{A}_{t+1}$ for all (a_t, y_t)
- Suppose we bound consumption choices from below: ensure c_{min} always affordable
- And use bounded support of income is $\mathbb{Y} = [y_{min}, y_{max}]$

Support and feasibility set in practice: support

- Upper bound of $\mathbb{A}_{t+1}$: maximum savings reached if $y_t = y_{max}$ and $c_t = c_{min}$ in the past

$$a_{t+1} \leq R^t a_1 + \sum_{\tau=1}^{t} R^\tau y_{max} - \sum_{\tau=1}^{t} R^\tau c_{min}$$

$$\Rightarrow \quad \text{UB}_{t+1} = R^t a_1 + R\frac{1-R^t}{1-R}(y_{max} - c_{min})$$

- Lower bound of $\mathbb{A}_{t+1}$: ensures c_{min} always affordable in future

$$a_{t+1} + \sum_{\tau=t+1}^{T} R^{(t+1)-\tau} y_{min} \geq \sum_{\tau=t+1}^{T} R^{(t+1)-\tau} c_{min}$$

$$\Rightarrow \quad \text{LB}_{t+1} = \frac{1-R^{t-T}}{1-R^{-1}}(c_{min} - y_{min})$$

- So $\mathbb{A}_{t+1} = [\text{LB}_{t+1}, \text{UB}_{t+1}]$

Support and feasibility set in practice: feasibility set

- Upper bound of $\mathcal{D}_t$ conditional on (a_t, y_t) ensures $c_t \geq c_{min}$

$$\begin{aligned} a_t + y_t - a_{t+1}R^{-1} &\geq c_{min} \\ \Rightarrow \quad \mathrm{UB}_{t+1}(a_t, y_t) &= R(a_t + y_t - c_{min}) \end{aligned}$$

- Lower bound of $\mathcal{D}_t$ equals lower bound of $\mathbb{A}_{t+1}$: LB_{t+1} can always be reached or otherwise problem has no solution

- So $\mathcal{D}_t(a_t, y_y) = [\mathrm{LB}_{t+1}, \mathrm{UB}_{t+1}(a_t, y_y)]$

Memoryless income process: optimality conditions

- Foc at time t: derivative of objective function at time t is zero

$$f_3\left(a_t, y_t, a_{t+1}\right) + \beta \int_{y \in \mathbb{Y}} \frac{\partial V_{t+1}\left(a_{t+1}, y\right)}{\partial a_{t+1}} \pi(y)\, dy = 0$$

- Work out marginal value of a_t:

$$\frac{\partial V_t\left(a_t, y_t\right)}{\partial a_t} = f_1 + \left[\underbrace{f_3 + \beta \int_{y \in \mathbb{Y}} \frac{\partial V_{t+1}}{\partial a_{t+1}} \pi(y)\, dy}_{=0}\right] \frac{\partial a_{t+1}}{\partial a_t} = f_1\left(a_t, y_t, a_{t+1}\right)$$

- So an interior optimum satisfies

$$\begin{aligned} f_3(a_t, y_t, a_{t+1}) + \beta \int_{y \in \mathbb{Y}} f_1(a_{t+1}, y, a_{t+2}) \pi(y)\, dy &= 0 \\ \Leftrightarrow \quad u'(c_t) - \beta R \mathrm{E}_t\left[u'(c_{t+1})\right] &= 0 \end{aligned}$$

Stochastic optimisation

Iid income process: Numerical solution

Computational algorithm

1. Parameterise model and select grids (A, Y) and compute weights π^j
2. Choose stopping criterion $\epsilon > 0$
3. Store $\mathrm{E}V_{T+1}\left(a^i_{t+1}\right) = 0$ for all $i = 1, \ldots, n_a$
4. Loop over t backwards: $t = T, \ldots, 1$

   Loop over $i = 1, \ldots, n_a$

   4.1 Compute for $j = 1, \ldots, n_y$

   $$g^{ij}_t = \arg\max_{a_{t+1} \in \mathcal{D}^{ij}_t} \left\{ u\left(a^i_t + y^j - \frac{a_{t+1}}{R}\right) + \beta \widetilde{\mathrm{E}V}_{t+1}\left(a_{t+1}\right) \right\}$$

   4.2 Compute the continuation value

   $$\mathrm{E}V^i_t = \sum_{j=1,\ldots,n_y} \left[u\left(a^i_t + y^j - \frac{g^{ij}_t}{R}\right) + \beta \widetilde{\mathrm{E}V}_{t+1}\left(g^{ij}_t\right) \right] \pi^j$$

Practical issues I

- State space is 2-dim: (a, y)
  - The income process could have a continuous support: discretise $\mathbb{Y}$ and solve problem in $n_a \times n_y$ points for each t
  - Bounds in $\mathbb{Y}$: ensure feasibility and measurability
- Grid in a to account for the many possible future circumstances
  - Feasibility amounts to ensure c_{min} remains affordable
  - Imposed on worst case scenario of future income so it holds under all possible future circumstances
- Continuation value: $\mathrm{E}_t V_{t+1}$
  - Measured at t conditional on existing information
  - Only argument in $\mathrm{E}_t V_{t+1}$ is a_{t+1}
  - Choice of grid in y to support integration
  - Need set of weights to calculate integral numerically, π^j

Practical issues II

We choose to store EV instead of V

- More efficient: saves computations in solution
- Can be used to recover V_t at (a, y)

$$V_t(a,y) = u\left(a + y - \frac{\widetilde{g}_t(a,y)}{R}\right) + \beta \widetilde{\mathrm{E}V}_{t+1}\left(\widetilde{g}_t(a,y)\right)$$

- If had stored V_t, step 4.1 would compute (for each (i, j, t))

$$g_t^{ij} = \underset{a_{t+1} \in \mathcal{D}_t^{ij}}{\arg\max} \left\{ u\left(a_t^i + y^j - \frac{a_{t+1}}{R}\right) + \beta \sum_{l=1}^{n_y} \widetilde{V}_{t+1}\left(a_{t+1}, y^l\right)\pi^l \right\}$$

involving n_y interpolations for each a_{t+1} called by maximisation routine

Numerical integration I

- Suppose we want to compute $\int_a^b f(y)\pi_y(y)dy$ where
  - π_y is the pdf of y
  - the value of f is known in points y^i in grid Y
- The numerical integral is a simple weighted average of f over a discrete selected grid

$$\int_a^b f(y)\pi_y(y)dy \simeq \sum_{i=1}^{n_y} f(y^i)w^i$$

- The simplest procedure (Tauchen)

1. Divide the distribution of y into n_y equal-probability intervals, Y^i
2. Compute the grid points $y^i = \mathrm{E}(y \mid Y^i)$
3. The weights are uniform: $w^i = n_y^{-1}$
4. Then $\int_{\mathbb{Y}} f(y)\pi_y(y)dy \simeq n_y^{-1} \sum_{i=1}^{n_y} f(y^i)$

Numerical integration II

Alternative procedures

- Gaussian quadrature: Gaussian nodes and weights $\{(y^i, w^i)\}$ are selected to make exact the numerical integral of polynomials of degree $2n_y + 1$ or less
  - Good option if f can be closely approximated by a polynomial
  - Weights and nodes depend on the distribution of y: Gauss-Laguerre for normal, Gauss-Hermite for log-normal, ...
- Monte-Carlo simulations: draw $\{y^i\}$ randomly from its distribution and compute simple average of $f(y)$ at random points

Practical issues III

- The algorithm we specified is implicitly designed to use with a search method
- But again it can be more efficient and accurate to use foc

Find root of Euler equation: CRRA utility

- At each (a_t^i, y^j, t) find root (a_{t+1}) of

$$u'\left(a_t^i + y^j - \frac{a_{t+1}}{R}\right) - \beta R \widetilde{dV}_{t+1}(a_{t+1}) = 0$$

- Inverse marginal utility reduces non-linearity in marginal value
- Can solve Euler equation in its *quasi-linearised* version

$$\left(a_t^i + y^j - \frac{a_{t+1}}{R}\right) - (\beta R)^{-\frac{1}{\gamma}} \widetilde{\mathrm{l}dV}_{t+1}(a_{t+1}) = 0$$

where the *quasi-linear* expected marginal value $(\mathrm{l}dV)$ is stored

$$\mathrm{l}dV_{t+1}^i = (u')^{-1}\left[dV_{t+1}^i\right] = \left[\sum_{j=1}^{n_y}\left(a_{t+1}^i + y^j - \frac{g_{t+1}^{ij}}{R}\right)^{-\gamma}\pi^j\right]^{-\frac{1}{\gamma}}$$

Stochastic optimisation

Autocorrelated income process

Autocorrelated income process

- More interesting model of income: AR(1) process
- We assume
$$\ln y_t = \alpha + \rho \ln y_{t-1} + e_t$$
- y_t is a Markov process: Markov structure of dynamic problem not compromised
- Stationarity requires that unconditional pdf of y is time-invariant
- Stationarity under log-normality requires $|\rho| < 1$ and, for all t
  - $\mathrm{E}(\ln y_t) = \alpha(1-\rho)^{-1}$
  - $\mathrm{Var}(\ln y_t) = \sigma_e^2 \left(1-\rho^2\right)^{-1}$

Autocorrelated income process: model

The consumption-savings problem is ($\mathcal{D}_t(a, y)$ as defined earlier)

$$
\begin{aligned}
& V_t(a_t, y_t) \\
= & \max_{a_{t+1} \in \mathcal{D}_t} \left\{ f\left(a_t, y_t, a_{t+1}\right) + \beta \int V_{t+1}\left(a_{t+1}, y_t^{\rho} \exp\left\{\alpha + e\right\}\right) dF_e(e) \right\}
\end{aligned}
$$

- Generally need to bound domain of e to ensure feasibility and measurability at all points
- The Euler equation is

$$
u'(c_t) = \beta R \mathrm{E}_t \left[u'(c_{t+1}) \mid y_t \right]
$$

Simple example I

Not most appealing 2-period model... but can be solved explicitly

- Period 1: consumer endowed with (a_1, y_1), consumes c_1
- Period 2:
  - $a_2 = R(a_1 + y_1 - c_1)$
  - $y_2 = \rho y_1 + e_2$
  - $c_2 = R(a_1 + y_1 - c_1) + (\rho y_1 + e_2)$

  where e_2 is a rv of mean zero, unknown from period 1 and unrelated to other model variables
- Utility function: $u(c) = \delta_0 + \delta_1 c + \delta_2 c^2$
- Consumers problem:

$$\max_{c_1} \left\{ u(c_1) + \beta \mathrm{E}_1 u \left[R(a_1 + y_1 - c_1) + (\rho y_1 + e_2) \right] \right\}$$

Simple example II

- The Euler equation is (with $\beta R = 1$)

$$\begin{aligned}
\delta_1 + \delta_2 c_1 &= \delta_1 + \delta_2 \mathrm{E}\left[R(a_1 + y_1 - c_1) + (\rho y_1 + e_2)\right] \\
&= \delta_1 + \delta_2 \left[R(a_1 + y_1 - c_1) + \rho y_1\right]
\end{aligned}$$

- With solution

$$c_1 = \frac{R}{1+R} a_1 + \frac{\rho + R}{1+R} y_1$$

- If $\rho = 0$: income shocks do not persist and consumption responds less to shocks
- If $\rho = 1$: permanent income shocks and consumption responds fully to shocks

Solution algorithm

1. Parameterise model and select grids (A, Y) and compute weights Q^{jl}
2. Choose stopping criterion $\epsilon > 0$
3. Store $\mathrm{E}V_{T+1}\left(a^i_{t+1}, y^j\right) = 0$ for all $i = 1, \ldots, n_a$ and $j = 1, \ldots, n_y$
4. Loop over t backwards: $t = T, \ldots, 1$

   Loop over $i = 1, \ldots, n_a$

   4.1 Compute for $j = 1, \ldots, n_y$

   $$g^{ij}_t = \underset{a_{t+1} \in \mathcal{D}^{ij}_t}{\arg\max} \left\{ u\left(a^i_t + y^j - \frac{a_{t+1}}{R}\right) + \beta \widetilde{\mathrm{E}V}_{t+1}\left(a_{t+1}, y^j\right)\right\}$$

   4.2 Compute the continuation value at point $(a_t, y_{t-1}) = (a^i_t, y^l)$

   $$\mathrm{E}V^{il}_t = \sum_{j=1,\ldots,n_y} \left[u\left(a^i_t + y^j - \frac{g^{ij}_t}{R}\right) + \beta \widetilde{\mathrm{E}V}_{t+1}\left(g^{ij}_t, y^j\right)\right] Q^{lj}$$

Practical issues

- The continuation value at time t is $\mathrm{E}_t\left[V_{t+1}(a_{t+1}, y_{t+1}) \mid y_t\right]$, a function of (a_{t+1}, y_t)
- If the foc were to be used in the solution, the linearised expected marginal value in time t Euler equation would also be a function of (a_{t+1}, y_t)
- Persistency in y_t implies that the integration weights Q need to be conditional on the past realisation of y

Transition function: simple procedure to determine Q^{ji}

- Consider a stationary Markov process

$$x_t \;=\; \alpha + \rho x_{t-1} + e_t \quad \text{where} \quad e \sim \mathcal{N}(0, \sigma^2)$$

- A simple procedure to compute Q^{jl}
  1. Divide the domain $\mathbb{X}$ in n_x intervals $\{X^i = [\underline{x}^i, \overline{x}^i]\}$
  2. Compute the grid points $x^i \;=\; \mathrm{E}(x^i \mid x^i \in X^i)$
  3. Then

$$\begin{aligned}
Q^{ji} &= \text{Prob}\left(x_t \in X^i \mid x_{t-1} = x^j\right) \\
&= \text{Prob}\left(\underline{x}^i \leq \alpha + \rho x^j + e_t \leq \overline{x}^i\right) \\
&= \text{Prob}\left(\underline{x}^i - \alpha - \rho x^j \leq e_t \leq \overline{x}^i - \alpha - \rho x^j\right) \\
&= \Phi\left(\frac{\overline{x}^i - \alpha - \rho x^j}{\sigma}\right) - \Phi\left(\frac{\underline{x}^i - \alpha - \rho x^j}{\sigma}\right)
\end{aligned}$$

Practical session 3

Infinite horizon

The problem

Consumption-savings with infinite horizon

- Often useful to consider dynamic problems in infinite horizon
  - Short time periods
  - End period very far away
  - End period uncertain and not becoming more likely over time
- Inherits many of the features of finite horizon problem but conceptually more complex
- Markov structure of problem is key: cannot deal with dependencies on infinite past
- Stationarity (at least in limit) is also crucial: dimensionality problem, and possibly measurement problems as well

The problem at time t

This is

$$V_t(a_t, y_t) = \mathrm{E}_t\left[\max_{\mathcal{D}_{t:\infty}(a_t,y_t)} \sum_{\tau=t}^{\infty} \beta^{\tau-t} f(a_\tau, y_\tau, a_{\tau+1}) \mid a_t, y_t\right]$$

- The horizon is always infinite, whichever t
  - Conditional on (a, y), the feasibility set is always the same, $\mathcal{D}_\infty(a, y)$
  - Conditional on (a, y), the problem is always the same, $V(a, y)$
- Given stationarity the infinite horizon problem is time-invariant
- Hence can drop time indexes

Recursive form I

The functional equation

$$V(a,y) = \mathrm{E}\left[\max_{\mathcal{D}_\infty(a,y)} \sum_{t=0}^{\infty} \beta^t f(a_t, y_t, a_{t+1}) \middle| a, y\right]$$

$$= \max_{a' \in \mathcal{D}(a,y)} \left\{ f(a,y,a') + \beta \underbrace{\mathrm{E}_{y'|y}\left(\mathrm{E}\left[\max_{\mathcal{D}_\infty(a',y')} \sum_{t=0}^{\infty} \beta^t f(a_t, y_t, a_{t+1}) \middle| a', y'\right]\right)}_{\text{Expected value today of } V' \text{ tomorrow, conditional on } (a,y)} \right\}$$

$$= \max_{a' \in \mathcal{D}(a,y)} \left\{ f(a,y,a') + \beta \mathrm{E}_{y'|y}\left[V(a',y')\right] \right\}$$

Recursive form II

$$V(a,y) = \max_{a' \in \mathcal{D}(a,y)} \left\{ f(a,y,a') + \beta \underbrace{\mathrm{E}_{y'|y}\left[V(a',y')\right]}_{\int_{\mathbb{Y}} V(a',y')Q(y,y')dy'} \right\}$$

- This is the Bellman equation
- The solution is a fixed point V of this functional equation
- Key to the specification: stationarity of the Markov process

Feasibility set

- Determined by a set of conditions

$$
\begin{aligned}
a' &= R(a + y - c) \\
\ln y' &= \alpha + \rho \ln y + e' \\
e &\sim \mathcal{N}\left(0, \sigma_e^2\right) \\
\ln y_0 &\sim \mathcal{N}\left(\mu_{\ln y}, \sigma_{\ln y}^2\right) \\
(a_0, y_0) &\in \mathbb{A} \times \mathbb{Y} \\
&\text{a bounding condition}
\end{aligned}
$$

- Stationarity requires

$$
\mu_{\ln y} = \frac{\alpha}{1 - \rho} \quad \text{and} \quad \sigma_{\ln y}^2 = \frac{\sigma_e^2}{1 - \rho^2}
$$

Bounding condition

- Typical assumption is that transversality conditions is satisfied

$$\lim_{t\to\infty} \beta^t \mathrm{E}\left[\frac{\partial f(a_t, y_t, a_{t+1})}{\partial a_t} a_t\right] = 0$$

- This is similar to the Kuhn-Tucker conditions
  - In the limit, either the present marginal value of assets is 0 or the agent consumes all her wealth
  - Ensures that consumer cannot borrow too much: present value of assets in far future is zero

Infinite horizon

Existence and uniqueness of solution

Contraction Mapping result I

- $C(X)$ is space of continuous functions with support X
- $T : C(X) \to C(X)$ is a transformation (mapping): $Tw(x) = v(x)$
- T satisfying *Blackwell sufficient conditions* is a contraction mapping

Monotonicity: if $v, w \in C(X)$ and $v(x) \leq w(x)$ for all $x \in X$, then $Tv(x) \leq Tw(x)$

Discounting: there is $\beta \in (0, 1)$ such that $T(v + k)(x) \leq Tv(x) + \beta k$ for all $k > 0$, where $(v + k)(x) = v(x) + k$

Contraction Mapping result II

- *Fixed point* of T is a function $v: \; T(v(x)) = v(x)$
- **Contraction Mapping Theorem**

  If T is a contraction mapping with modulus β, then
  1. T has exactly 1 fixed point
  2. Fixed point can be reached iteratively from any $v_0 \in C(X)$
- Bellman equation defines a contraction mapping with modulus β

$$TV(a,y) = \max_{a' \in \mathcal{D}(a,y)} \left\{ f(a,y,a') + \beta \int_{\mathbb{Y}} V(a',y')Q(y,y')dy' \right\}$$

Existence and uniqueness result

The dynamic optimisation problem is

$$V(a, y) = \max_{a' \in \mathcal{D}(a,y)} \left\{ f(a, y, a') + \beta \int_{\mathbb{Y}} V(a', y') Q(y, y') dy' \right\}$$

- $\beta \in (0, 1)$;
- f: real-valued, continuous, strictly concave in a and bounded;
- y: Markov process in the compact set $\mathbb{Y}$;
- $\mathcal{D}(a, y)$: non-empty, compact and convex.

Then:

1. there exists a unique function V that solves this problem;
2. V is continuous and strictly concave in a;
3. $g(a, y)$ exists and is a (unique) continuous, single-valued function.

Other properties of the problem

Other properties of f are transferred to V through the mapping T:

1. f also $\mathcal{C}^1$ in $(a, a') \in \text{int}(\mathbb{A})^2$ and $g(a, y) \in \text{int}(\mathcal{D}(a, y))$

   $\Rightarrow$ V is $\mathcal{C}^1$ in a and $V_1(a, y) = f_1(a, y, g(a, y))$

2. f also strictly increasing in a and $\mathcal{D}(a, y) \geq \mathcal{D}(a', y)$ for $a \geq a'$

   $\Rightarrow$ V is strictly increasing in a

3. 2 also true for y if f and $\mathcal{D}(a, y)$ are strictly increasing in y

Optimality conditions for interior solution

- Euler equation under the continuous differentiability conditions

$$\begin{aligned} f_3(a,y,a') + \beta \int_{\mathbb{Y}} f_1(a',y',g(a',y'))Q(y,y')dy' &= 0 \\ \Leftrightarrow \quad f_3(a,y,a') + \beta \mathrm{E}_{y'|y}\left[f_1(a',y',g(a',y'))\right] &= 0 \end{aligned}$$

  Or in terms of the utility function

$$\frac{d\,u(c)}{d\,c} = \beta R \mathrm{E}_{y'|y}\left[\frac{d\,u(c')}{d\,c'}\right]$$

- Euler and transversality conditions: necessary and sufficient for the interior optimum $a' = g(a,y)$

$$\lim_{t\to\infty} \beta^t \mathrm{E}\left[\frac{\partial f(a_t, y_t, a_{t+1})}{\partial a_t} a_t\right] = 0$$

Infinite horizon

Simple example

Simple example I

- Consider the problem

$$V(a) \quad = \quad \max_{c>0} \{\ln(c) + \beta V(R(a-c))\}$$

- The Euler equation is

$$\frac{1}{c_t} = \beta R \frac{1}{c_{t+1}} \quad \Leftrightarrow \quad c_{t+1} = \beta R c_t = (\beta R)^t c_0$$

- The transversality condition is

$$\lim_{t\to\infty} \beta^t u'(c_t) a_t \quad = \quad \lim_{t\to\infty} \frac{\beta^t a_t}{(\beta R)^t c_0} \quad = \quad \lim_{t\to\infty} \frac{a_t}{R^t c_0} \quad = \quad 0$$

Simple example II

Work out the value of a_t

$$
\begin{aligned}
a_t &= R(a_{t-1} - c_{t-1}) \\
&= R(R[a_{t-2} - c_{t-2}] - c_{t_1}) \ldots \\
&= R^t a_0 - \sum_{\tau=0}^{t-1} R^{t-\tau} c_\tau \\
&= R^t a_0 - \sum_{\tau=0}^{t-1} R^{t-\tau} (\beta R)^\tau c_0 \\
&= R^t a_0 - R^t c_0 \sum_{\tau=0}^{t-1} \beta^\tau \\
&= R^t \left(a_0 - c_0 \frac{1-\beta^t}{1-\beta} \right)
\end{aligned}
$$

Simple example III

- We got $a_t \;=\; R^t \left(a_0 - c_0 \frac{1-\beta^t}{1-\beta}\right)$
- Replace in transversality condition to yield

$$\lim_{t\to\infty} \frac{a_t}{R^t c_0} \;=\; \lim_{t\to\infty} \frac{R^t \left(a_0 - c_0 \frac{1-\beta^t}{1-\beta}\right)}{R^t c_0}$$
$$= a_0 - c_0 \frac{1}{1-\beta} \;=\; 0$$

- Hence the solution is $c_0 = a_0(1-\beta)$
- More generally, $c_t = a_0(1-\beta)(\beta R)^t$

Infinite horizon

Numerical solution

Recursion

$$V(a,y) = \max_{a' \in \mathcal{D}(a,y)} \left\{ f(a,y,a') + \beta \mathrm{E}_{y'|y} \left[V(a',y') | \, a,y \right] \right\}$$

- *Contraction Mapping Theorem*
  1. The problem has a unique fixed point V
  2. It can be reached iteratively from any starting function V_0

- Value function iteration
  1. find optimal savings

$$g_n(a,y) = \arg\max_{a' \in \mathcal{D}(a,y)} \left\{ f(a,y,a') + \beta \mathrm{E}_{y'|y} \left[V_{n-1}(a',y') \right] \right\}$$

  2. compute new value function

$$V_n(a,y) = f(a,y,g_n(a,y)) + \beta \mathrm{E}_{y'|y} \left[V_{n-1}(g_n(a,y),y') \right]$$

Solution algorithm: value function iteration

1. Parameterise model and select grids (A, Y) and compute weights Q^{jl}
2. Choose stopping criterion $\epsilon > 0$
3. Select initial guess $\mathrm{E}V_0\left(a^i, y^j\right)$ for all $\left(a^i, y^j\right) \in A \times Y$
4. Iterate until convergence, for $n = 1, \ldots$

   4.1 For all a^i in grid A compute

   $$(a) \quad g_n^{ij} = \arg\max_{a' \in \mathcal{D}^{ij}} \left\{ f\left(a^i, y^j, a'\right) + \beta \widetilde{\mathrm{E}V}_{n-1}\left(a', y^j\right) \right\} \quad \text{for all } y = y^j$$

   $$(b) \quad \mathrm{E}V_n^{il} = \sum_{j=1}^{n_y} \left[f\left(a^i, y^j, g_n^{ij}\right) + \beta \widetilde{\mathrm{E}V}_{n-1}\left(g_n^{ij}, y^j\right) \right] Q^{lj} \quad \text{for all } y_{-1} = y^l$$

   4.2 Check distance between $\mathrm{E}V_{n-1}$ and $\mathrm{E}V_n$

   - If larger than ϵ then go back to step 4.1
   - Else accept solution (g_n, V_n) and stop

Practical issues

- Time subscript dropped and loop is now until convergence of V to fixed point
- Initial guess
  - should be $\mathcal{C}^1$
  - could be $\mathrm{E}V_0 = 0$: implying consumer saves nothing to next period
  - better solution is $\mathrm{E}V_0 = u(c)$
- Solution using Euler equation: store dV, a function of (a, y_{-1})
- Distance in continuation value: max absolute difference (levels/relative)

Feasibility set

- Transversality condition not practical

$$\lim_{t\to\infty} \beta^t \mathrm{E}\left[f_1(a_t, y_t, a'_{t+1})a_t\right] = 0$$

- Implication: consumer avoids low assets, where f_1 arbitrarily large
- $\Rightarrow$ ensure c_{min} always affordable in worst possible scenario

$$a + \sum_{t=0}^{\infty} R^{-t} y_{min} \geq \sum_{t=0}^{\infty} R^{-t} c_{min} \quad \Leftrightarrow \quad a + \frac{1}{1-R^{-1}}\left(y_{min} - c_{min}\right) \geq 0$$

- If present state is (a, y) $\Rightarrow$ optimal savings a' must lie in interval

$$\mathcal{D}(a, y) = \left[-\frac{1}{1-R^{-1}}\left(y_{min} - c_{min}\right),\ R\left(a + y - c_{min}\right)\right]$$

Now for the final practical example!